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A NECESSARY CONDITION FOR AN ELLIPTIC ELEMENT TO BELONG TO A UNIFORM TREE LATTICE

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ABSTRACT. Let X be a universal cover of a finite connected graph, $G = \mathrm{Aut}(X)$, and Γ a group acting discretely and cocompactly on X, i.e., a uniform lattice on X. We give a necessary condition for an elliptic element of G to belong to a uniform lattice or to the commensurability group. By using this condition, we construct some explicit examples.

Continuing the classic Bass-Serre theory on graphs of groups [S], Bass developed the covering theory for graphs of groups [B]. Using this, Bass and Kulkarni developed the uniform tree lattices theory in their joint paper [BK]. In that paper they obtained a lot of important results. It is fruitful to think of (G, X, Γ) as a combinatorical analogue of $(\mathrm{PSL}_2(\mathbf{R})$, upper half plane, fuchsian group).

Let X be a 'uniform tree', i.e., the universal cover of a finite connected graph, $G = \mathrm{Aut}(X)$, equipped with compact open topology. The subgroup $H < G$ is discrete iff every vertex stabilizer H_x for $x \in VX$ is finite, where VX is the set of all vertices of X. We call $\Gamma < G$ a uniform X-lattice if Γ is discrete and the quotient graph $\Gamma\backslash X$ is finite (i.e., VX has only finitely many Γ-orbits). Let Γ_0, Γ_1 be subgroups of G. Γ_0 and Γ_1 are said to be commensurable (denoted $\Gamma_0 \sim \Gamma_1$), if the index $[\Gamma_i : \Gamma_0 \cap \Gamma_1]$ is finite for $i = 0, 1$. The commensurator (or "virtual normalizer') of Γ in G is the group $C_G(\Gamma) = \{g \in G | g\Gamma g^{-1} \sim \Gamma\}$. It was shown in [BK] that, up to G-conjugacy, any two uniform lattices in G are commensurable. Thus the commensurator $C_G(\Gamma)$ of a uniform lattice $\Gamma \leq G$ is, up to conjugacy, independent of Γ; we denote it by $C(X)$. It is proved in [L1] that $C(X)$ is dense in G, which was conjectured in [BK].

In this paper, we give a necessary condition for an elliptic element (i.e., one having fixed points) of G to belong to a uniform lattice or to the commensurability group $C(X)$. By this condition, it is then easy to construct some automorphisms of X which do not belong to a uniform lattice, nor do they belong to $C(X)$.

We address here the following questions:

Question. Let X be a uniform tree, $G = \mathrm{Aut}(X)$, $g \in G$. When is there a uniform X-lattice Γ: (a) such that $g \in \Gamma$?; (b) such that $g \in C_G(\Gamma)$?

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We begin by quoting

Proposition 1 [BK, (4.2) Conjugacy Theorem]. *If g is hyperbolic (i.e., acting on X freely), then g belongs to a uniform lattice.*

So the case of main interest is when g is of finite order. The following notion, due to Gelfand, will be useful for our discussion.

Definition. Let G be a locally compact group. An element $u \in G$ will be called G-unipotent if the closure of its G-conjugacy class $C_G(u)$ contains 1, where $C_G(u) = \{gug^{-1} | g \in G\}$.

Lemma 1. *Assume that $\Gamma \backslash G$ is compact, in the sense that $G = K \cdot \Gamma$ for some compact set $K \subset G$. If $\sigma \in G$ is G-unipotent then the closure of its Γ-conjugacy class $C_\Gamma(\sigma)$ contains 1, where $C_\Gamma(\sigma) = \{\gamma\sigma\gamma^{-1} | \gamma \in \Gamma\}$.*

Proof. Say

$$1 = \lim_n g_n \sigma g_n^{-1}, \qquad g_n \in G, \ n = 1, 2, \dots.$$

Write $g_n = k_n \gamma_n$, where $k_n \in K$, $\gamma_n \in \Gamma$.

Passing to a subsequence we can (compactness of K) assume that $k_n \to k$ for some $k \in K$. Then

$$1 = \lim_n k_n \gamma_n \sigma \gamma_n^{-1} k_n^{-1} = \lim_n k \gamma_n \sigma \gamma_n^{-1} k^{-1},$$

and so

$$1 = \lim_n \gamma_n \sigma \gamma_n^{-1}, \qquad \gamma_n \in \Gamma. \quad \text{Q.E.D}$$

Proposition 2. *Let $\Gamma \in \mathrm{Lat}_u(X)$; then $C_G(\Gamma)$, in particular Γ, contains no G-unipotent element $\neq 1$.*

Proof. Suppose that $\sigma \in C_G(\Gamma)$ is G-unipotent.

Put $\Gamma' = \Gamma \cap \sigma\Gamma\sigma^{-1}$, a subgroup of finite index in Γ. Applying Lemma 1 to Γ', we have $1 = \lim_n \gamma_n \sigma \gamma_n^{-1}$ with $\gamma_n \in \Gamma'$. Hence,

$$\sigma^{-1} = \lim_n \sigma^{-1} \gamma_n \sigma \gamma_n^{-1}.$$

But, for each n, $(\sigma^{-1}\gamma_n\sigma)\gamma_n^{-1} \in \langle \sigma^{-1}\Gamma'\sigma, \Gamma' \rangle \leq \Gamma$, and Γ is discrete. Hence, for $n \gg 0$,

$$\sigma^{-1} = \sigma^{-1}\gamma_n\sigma\gamma_n^{-1};$$

whence, $\gamma_n\sigma\gamma_n^{-1} = 1$, i.e., $\sigma = 1$. Q.E.D.

Thus we get a necessary condition for an elliptic $g \neq 1$ to belong to a uniform tree lattice or $C(X)$ that g is not a G-unipotent element.

Lemma 2. *An element $\sigma \in G$ is G-unipotent iff it is elliptic and its tree of fixed points contains a G-translate of any given finite subtree.*

Proof. Assume that $\sigma \in G$ is G-unipotent. Thus, by the definition, there is a sequence $\{g_n \in G, \ n = 1, 2, \dots\}$ such that $\lim_n g_n^{-1}\sigma g_n = 1$. In other words, for any given finite subtree Y of X and for $n \gg 0$, we have $g_n^{-1}\sigma g_n | Y = \mathrm{id} | Y$, i.e., $\sigma | (g_n Y) = \mathrm{id} | (g_n Y)$. So, σ is elliptic and its tree of fixed points contains $g_n Y$, where $g_n \in G$ and Y is any given finite subtree of X.

Conversely, suppose that σ is elliptic and its tree of fixed points contains a G-translate of any given finite subtree.

For $a \in VX$, put $B_a(n) = \{x \in VX | d(a, x) \leq n\}$. Then $\{B_a(n),\ n = 1, 2, \ldots\}$ is a sequence of finite subtrees of X. For each $B_a(n)$, by the assumption, there is $g_n \in G$, such that

$$\sigma|(g_n B_a(n)) = \mathrm{id}\,|(g_n B_a(n)),$$

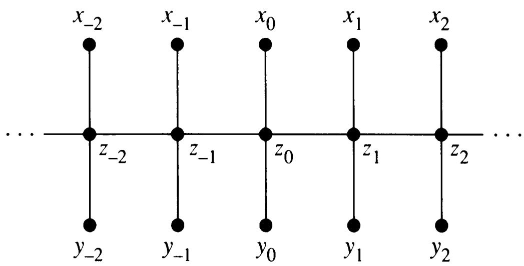
i.e.,

$$g_n^{-1}\sigma g_n | B_a(n) = \mathrm{id}\,|B_a(n), \qquad g_n \in G,\ n = 1, 2, \ldots.$$

So, $\lim_n g_n^{-1}\sigma g_n = 1$, $g_n \in G$. Q.E.D.

Now, it is easy to construct G-unipotent elements of finite order, which thus lie in no uniform lattice (or even the commensurator of one).

Example 1. Let X be the following virtually linear tree:



Clearly, X is a uniform tree. In fact, let $g \in \mathrm{Aut}(X)$ be defined by

$$g(x_n) = x_{n+1}, \quad g(y_n) = y_{n+1}, \quad g(z_n) = z_{n+1}, \qquad n = 0, \pm 1, \pm 2, \ldots,$$

then $\langle g \rangle$ is a uniform lattice of X: $\langle g \rangle$ is discrete and $\langle g \rangle \backslash X$ is finite.

Define $\sigma \in G = \mathrm{Aut}(X)$, such that $\sigma(x_0) = y_0$, $\sigma(y_0) = x_0$, and σ acts on $X - \{x_0, y_0\}$ trivially.

Clearly, the subtree of fixed points of σ contains a G-translate of any given finite subtree of X. By Lemma 2, σ is a nontrivial G-unipotent. Hence, by Proposition 2, σ does not belong to any uniform lattice nor even to the commensurator of any uniform lattice.

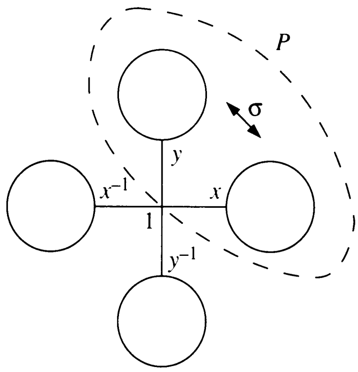
Example 2. Let X be the Cayley tree $\mathrm{Cay}(F(x, y), \{x, y\})$, where $F(x, y)$ is a free group on a basis $\{x, y\}$. Let $\alpha \in \mathrm{Aut}(F(x, y))$, such that $\alpha(x) = y$, $\alpha(y) = x$. Put

$$P = \{u \in F(x, y) | \text{reduced word of } u \text{ begins with } x \text{ or } y\}.$$

Note that α defines an automorphism of X and $\alpha P = P$. Define $\sigma \in \mathrm{Aut}(X)$ by

$$\sigma(u) = \begin{cases} \alpha(u) & \text{if } u \in P, \\ u & \text{if } u \notin P. \end{cases}$$

$X = \mathrm{Cay}(F(x, y), x, y)$:



Since σ switches two branches of X and fixes the other two branches, the subtree of fixed points of σ contains G-translate of any given finite subtree of X. By Lemma 2, σ is a nontrivial G-unipotent. Hence, σ lies in no uniform lattice nor even the commensurator of one.

On the other hand, we have

Proposition 3. *Let $\Gamma \leq G$ be a uniform lattice and $F \leq C_G(\Gamma)$ a subgroup such that $F \cdot \Gamma/\Gamma$ is finite. Then $F \leq \Gamma'$ for some $\Gamma' \sim \Gamma$.*

Proof. We may assume that $F \cdot \Gamma = S \cdot \Gamma$, where S is a finite subset of $C_G(\Gamma)$. Put

$$\Gamma_0 = \bigcap_{g \in F \cdot \Gamma} g\Gamma g^{-1} = \bigcap_{s \in S} s\Gamma s^{-1}.$$

As $s \in C_G(\Gamma)$, $s\Gamma s^{-1} \sim \Gamma$ for each $s \in S$. Since the intersection of two subgroups of finite index has finite index, it follows that a commensurability class of subgroups of G is stable under finite intersection. Thus the finite intersection Γ_0 is commensurable with Γ. And, clearly, Γ_0 is normalized by F, i.e., $F \leq N_G(\Gamma_0)$. According to [BK, Corollary (6.4)], $\Gamma_0 \backslash N_G(\Gamma_0)$ is finite, so $N_G(\Gamma_0) \sim \Gamma_0 \sim \Gamma$. Thus the proposition is proved by taking $\Gamma' = N_G(\Gamma_0)$.

Remark. Proposition 3 applies notably when $F \leq C_G(\Gamma)$ is finite or when $F = \langle g \rangle$ with $g^n \in \Gamma$ for some $n > 0$.

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