ON QUASI-CONTINUOUS RINGS

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(Communicated by Lance W. Small)

ABSTRACT. A well-known result of Utumi asserts that a two-sided continuous two-sided artinian ring is quasi-Frobenius. In this paper we extend Utumi's result to quasi-continuous rings.

INTRODUCTION AND DEFINITIONS

A well-known result of Utumi [7] asserts that a two-sided continuous two-sided artinian ring is quasi-Frobenius. In [4] this result was extended to two-sided continuous rings with ACC on essential left and right ideals. In [3], motivated by a result of Carl Faith on self-injective rings, it was shown that a two-sided continuous ring with ACC on left annihilators is quasi-Frobenius. There are examples of two-sided artinian one-sided continuous rings which are not quasi-Frobenius (see [4]). In this paper we extend Utumi's result to quasi-continuous rings.

Throughout this paper all rings considered are associative with identity and all modules are unitary $R$-modules. We write $J(M)$, $Z(M)$, $Soc(M)$, and $E(M)$ for the Jacobson radical, the singular submodule, the socle, and the injective hull of $rM$, respectively. For any subset $X$ of $R$, $l_R(X)$ represents the left annihilator of $X$ in $R$.

Consider the following conditions on a module $rM$:

(C1) Every submodule of $M$ is essential in a summand of $M$.

(C2) Every submodule isomorphic to a summand of $M$ is itself a summand.

(C3) If $M_1$ and $M_2$ are summands of $M$ with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of $M$.

$M$ is called continuous if it satisfies conditions (C1) and (C2), quasi-continuous if it satisfies (C1) and (C3), and a CS-module if it satisfies condition (C1) only.

It is easy to see that (C2) implies (C3), but the converse is not true in general. Thus, every continuous module is quasi-continuous. The ring of integers $\mathbb{Z}$ is...
an example of a commutative, noetherian, quasi-continuous ring which is not continuous (and hence not quasi-Frobenius). For a full account of the subject of continuous and quasi-continuous modules we refer the reader to [5].

The result

Theorem 1. A left artinian two-sided quasi-continuous ring is quasi-Frobenius.

Before we begin the proof we need the following lemmas.

Lemma 1. \( \bigoplus_{i=1}^{n} M_i \) is quasi-continuous if and only if each \( M_i \) is quasi-continuous and \( M_j \)-injective for all \( j \neq i \).

Proof. See [5, Corollary 2.14].

Lemma 2. Let \( M_1 \) and \( M_2 \) be summands of a quasi-continuous module \( M \). If \( E(M_1) \cong E(M_2) \), then \( M_1 \cong M_2 \).

Proof. See [5, Theorem 2.31].

Lemma 3. Let \( R \) be a left quasi-continuous ring. Then \( R \) is left continuous if and only if \( Z(R \cdot R) = J(R) \) and \( R/J(R) \) is regular.

Proof. See [5, Proposition 3.15].

Lemma 4. Let \( R \) be a two-sided continuous ring with ACC on left annihilators. Then \( R \) is a quasi-Frobenius ring.

Proof. See [3, Theorem 1].

Lemma 5. Let \( R \) be a semiperfect ring. Then \( Z(R \cdot R) \subseteq J(R) \) and \( Z(R/R) \subseteq J(R) \).

Proof. Since \( R \) is semiperfect, we can write \( R = R \cdot e \oplus R(1 - e) \) such that \( R \cdot e \subseteq Z(R \cdot R) \), \( Z(R \cdot R) \cap R(1 - e) \) is small in \( R \), and \( e^2 = e \in R \). Since \( Z(R \cdot R) \) does not contain nonzero idempotents, \( Z(R \cdot R) \) is small in \( R \). Thus \( Z(R \cdot R) \subseteq J(R) \). Similarly \( Z(R/R) \subseteq J(R) \).

The next lemma is a key result for the proof of Theorem 1.

Lemma 6. Let \( R \) be a semiprimary left quasi-continuous ring. Then \( R \) is left continuous.

Proof. Since \( R \) is semiperfect, we can write \( R = \bigoplus_{i=1}^{n} R \cdot e_i \) a direct sum of principal indecomposable left ideals \( R \cdot e_i \). Each \( e_i \) is a nonzero primitive idempotent. By Lemma 1, each \( R \cdot e_i \) is quasi-continuous. Since an indecomposable CS-module is necessarily uniform, each \( R \cdot e_i \) is uniform. Since \( R \) is semiprimary, \( \text{Soc}(R \cdot e_i) \) is essential in \( R \cdot e_i \), and hence each \( R \cdot e_i \) has a unique minimal left ideal.

Without loss of generality we may assume that \( \mathcal{E} = \{e_1, \ldots, e_m\} \) is a basic set of primitive idempotents for \( R \). Let \( T_i = \text{Soc}(R \cdot e_i) \), \( 1 \leq i \leq m \). Hence \( E(T_i) = E(R \cdot e_i) \).

Claim. \( T_i \cdot e_k = 0 \), for every \( i \) and \( k \), \( 1 \leq i, k \leq m \).

Suppose \( k \neq i \) and \( T_i \cdot a e_k \neq 0 \) for some \( a \in J \). It is easy to see that \( T_i \cong T_i \cdot a e_k \) as minimal left ideals. Since \( T_i \cdot a e_k \subseteq R \cdot e_k \), we infer that \( T_k \cong T_k \cdot a e_k \).

Since \( T_i \cong T_k \), \( E(R \cdot e_i) \cong E(R \cdot e_k) \). Now an application of Lemma 2 ensures that \( R \cdot e_i \cong R \cdot e_k \), and hence \( i = k \), a contradiction. Now, if \( k = i \) and \( T_i \cdot a e_i \neq 0 \) for some \( a \in J \), then \( T_i \cdot a e_i \cong T_i \), and hence \( T_i = T_i \cdot a e_i \) whence \( T_i \cdot e_i = T_i \).
Let $e = e_1 + \cdots + e_m$ be the basic idempotent. Then $T_i Je = T_i Je_i = T_i$. Thus $T_i J = T_i R$. Since $R$ is semiprimary, $J^{n-1} \neq 0$ and $J^n = 0$ for some $n \geq 1$. Thus $(T_i J)(J^{n-1}) = (T_i R)(J^{n-1})$, and hence $T_i J^{n-1} = 0$. By repeating the argument, if necessary, we get $T_i J = 0$. Thus $T_i R = 0$, and hence $T_i = 0$, a contradiction. Thus $T_i J e_k = 0$ in every case, proving the claim.

Since $T_i Je_k = 0$ and $T_i Je = 0$, we infer that $T_i \subseteq l_R(J)$, $1 \leq i \leq m$. Since $R/J$ is semisimple, $l_R(J) = \text{Soc}(R_R)$, and hence $\text{Soc}(R_R) \subseteq \text{Soc}(R_R)$.

Now since $\text{Soc}_R R$ is essential in $R R$ and $\text{Soc}(R_R) \subseteq l(R)$, it follows that $J(R) \subseteq Z(R_R)$. By Lemma 5, we get $J(R) = Z(R_R)$. And, by Lemma 3, $R$ is left continuous.

Now the proof of Theorem 1 is an easy consequence of Lemma 6.

**Proof of Theorem 1.** By Lemma 6, $R$ is a two-sided continuous ring, and, by Lemma 4, $R$ is a quasi-Frobenius ring.

**Corollary 1.** Let $R$ be a left quasi-continuous ring with DCC on essential left ideals. Then $R$ is a left continuous ring.

**Proof.** By [1, Proposition 1.1], $R/\text{Soc}(R_R)$ is left artinian. And, by [2, Corollary 6], $R$ is left artinian. Then it follows from Lemma 6 that $R$ is left continuous.

**Corollary 2.** Let $R$ be a two-sided quasi-continuous ring with DCC on essential left ideals. Then $R$ is quasi-Frobenius.

**Remark.** There is an example of a commutative, local, semiprimary, continuous ring which is not injective (see Rizvi [6]).

## References


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