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ON SUPPORTLESS CONVEX SETS IN INCOMPLETE NORMED SPACES

V. P. FONF

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ABSTRACT. It is proved that every incomplete separable normed space M contains a closed bounded convex set W such that the closed linear span of W coincides with M and W contains no weakly supported points. This theorem answers a question of Klee and a question of Borwein and Tingley.

Let A be a subset of normed space X. A point $x \in A$ is called a support point of A if there exists a nonzero continuous linear functional $f \in X^*$ such that $f(x) = \sup f(A)$. We shall call a point $x \in A$ a weakly supported point if there exists a linear functional f such that the restriction f/A is nonzero and continuous and $f(x) = \sup f(A)$. The last definition was inspired by the paper of Klee [1]. The set of all support points of the set A (respectively weakly supported points) we denote by $\operatorname{supp} A$ (respectively $\operatorname{wsupp} A$). In 1958 Klee [2] gave a striking example of a convex bounded closed subset A in a certain incomplete normed space X (actually X is a dense subspace of ℓ_2) possessing the property $\operatorname{supp} A = \varnothing$ (a supportless set). Subsequently, Bishop and Phelps [3] proved that for every closed bounded convex subset A (a CBC subset) of the Banach space X the set of all support functionals $\sum(A)$ of the set A is dense in dual space X^*. Later in [4] they proved that for every CBC subset A of a Banach space the set $\operatorname{supp} A$ is dense in the boundary of the set A. Hence supportless CBC subsets may occur only in incomplete normed spaces. In 1985 Borwein and Tingley [5], developing ideas of Klee, constructed in every separable Banach space a dense subspace possessing a supportless CBC subset (additionally, their set is absorbing; see the paper [6] of the author where he got a characterization of incomplete normed spaces possessing supportless absorbing CBC subsets). The main conjecture (formulated in paper [5] which inspired our investigation) is that every incomplete normed space contains a supportless CBC subset. We shall prove this conjecture (actually in a stronger form, which permits us to answer a question of Klee concerning weakly supported points [1]).

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First, we need an auxiliary lemma. But let us remember that a system $\{x_i\}$ in a Banach space X is called an M-basis of the space X if $\{x_i\}$ is the minimal complete system with total biorthogonal system [7].

Lemma. *Let M be a dense proper subspace of the separable Banach space X. Then there exist an M-basis $\{x_i\}_1^\infty \subset M$ of the space X such that $\|x_i\| \leq 2^{-i}$, $i = 1, 2, \dots$, and a sequence $\{\xi_i\}_1^\infty$, $\sum|\xi_i| = 1$, such that $\sum \xi_i x_i \notin M$.*

Proof. Subsequently using an existense of an M-basis in every separable Banach space [8], the Krein-Milman-Rutman stability theorem [9], and density of M in the space X we can choose an M-basis $\{x_i\}$ of the space X in the subspace M. In addition we suppose that $\|x_i\| \leq 2^{-i-1}$ for $i = 1, 2, \dots$. If there exists a sequence $\{\xi_i\}$ with $\sum|\xi_i| = 1$ such that $\sum \xi_i x_i \notin M$ the proof is ended. Otherwise (i.e., for every $\{\eta_i\}$ with $\sum|\eta_i| = 1$ we have $\sum \eta_i x_i \in M$) let $\{\varepsilon_i\}$ be the decreasing sequence of stability of the M-basis $\{x_i\}$ [9]. Without loss of generality we can assume that $0 < \varepsilon_i < 2^{-i-1}$, $i = 1, 2, \dots$. Let $y \in X \backslash M$, $\|y\| \leq \varepsilon_1^2/2$. Hence by density of M in X there exists a sequence $\{z_i\}_0^\infty$, $z_0 = 0$, such that $\|y - z_i\| \leq \varepsilon_{i+1}^2/2$, $i = 1, 2, \dots$. We have $\|z_i - z_{i-1}\| \leq \varepsilon_i^2$, $i = 1, 2, \dots$. Let $x_i' = x_i + 1/\varepsilon_i(z_i - z_{i-1})$, $i = 1, 2, \dots$. Since $\|1/\varepsilon_i(z_i - z_{i-1})\| \leq \varepsilon_i$, it follows that $\{x_i'\}$ is an M-basis of the space X. Thus we have

$$\begin{aligned} y' &= \sum \frac{\varepsilon_i}{\sum \varepsilon_j} x_i' = \frac{1}{\sum \varepsilon_j} \sum \varepsilon_i x_i + \frac{1}{\sum \varepsilon_j} \sum (z_i - z_{i-1}) \\ &= \frac{1}{\sum \varepsilon_j} \sum \varepsilon_i x_i + \frac{1}{\sum \varepsilon_j} y. \end{aligned}$$

Since $\sum \varepsilon_i x_i / \sum \varepsilon_j \in M$ (by assumption) and $y \notin M$, it follows that $y' \notin M$. Setting $\xi_i = \varepsilon_i / \sum \varepsilon_j$, $i = 1, 2, \dots$, we complete the proof. □

The following theorem gives an answer to one of the main open questions in the paper [5] by Borwein and Tingley.

Theorem. *Every incomplete separable normed space M contains a closed bounded convex subset W such that the closed linear span of W coincides with M and W contains no weakly supported points, i.e.,*

$$\operatorname{wsupp} W = \varnothing.$$

Proof. Let X be the completion of M and $\{x_i\} \subset M$ be an M-basis of the space X possessing the properties that (see the lemma) $\|x_i\| \leq 2^{-i}$ for $i = 1, 2, \dots$ and there exists a sequence $\{\xi_i\}$ such that $\sum|\xi_i| = 1$ and $y = \sum \xi_i x_i \notin M$. Let us define an operator $T : \ell_1 \to X$ by $Te_i = x_i$, $i = 1, 2, \dots$, where $\{e_i\}$ is the canonical basis of the space ℓ_1. So T is an injective compact operator, and the image $V = TU(\ell_1)$ of the unit ball $U(\ell_1)$ of the space ℓ_1 is closed. Let $W = M \cap (-\frac{1}{2}y + V)$, and introduce the following notation:

$$L = T^{-1}(M), \quad z = T^{-1}y = \sum \xi_i e_i, \quad A = T^{-1}(W) = L \cap (-\tfrac{1}{2}z + U(\ell_1)).$$

By density of the set $\operatorname{lin}\{e_i\}$ in the space ℓ_1 we have

$$\operatorname{cl}(A \cap \operatorname{lin}\{e_i\}) = -\tfrac{1}{2}z + U(\ell_1). \tag{1}$$

Therefore,

$$\operatorname{cl}\operatorname{lin} A = l_1. \tag{2}$$

Hence

$$\operatorname{cl}\operatorname{lin} W = \operatorname{cl}\operatorname{lin} TA = X \tag{3}$$

and

$$\operatorname{cl}(W \cap \operatorname{lin}\{x_i\}) \supset W. \tag{4}$$

Now suppose that a linear functional f on M exists such that the restriction f/W is continuous and nonzero and f attains its supremum on the set W at some point $x_0 \in W : f(x_0) = \sup f(W)$. Without loss of generality we can assume that the functional f is defined on the whole space X. Denote by $\hat{f}$ the linear functional on the space ℓ_1 defined by $\hat{f}(u) = f(Tu)$, $u \in \ell_1$. It is easily verified that $\hat{f}/A$ is w^*-continuous in the duality $c_0^* = \ell_1$ (remember that f/W is continuous, and note that the operator $T/A : (A, w^*) \to W$ is continuous as the restriction of an adjoint operator). Since $\frac{1}{2}e_i = -\frac{1}{2}z + (\frac{1}{2}z + \frac{1}{2}e_i) \in A$ and $w^*\text{-}\lim e_i = 0$, $\lim \hat{f}(e_i) = 0$. As f/W is a nonzero continuous functional, (4) gives the existence of an integer j such that $f(x_j) \neq 0$, so $\hat{f}(e_j) \neq 0$. Now let h be the element of the space c_0 such that $h(e_i) = \hat{f}(e_i)$, $i = 1, 2, \ldots$. Obviously, $h \neq 0$, and

$$h/\operatorname{lin}\{e_i\} = \hat{f}/\operatorname{lin}\{e_i\}. \tag{5}$$

Moreover, $h/A = \hat{f}/A$. Indeed let $v \in A, v = -\frac{1}{2}z + \sum \gamma_i e_i$, $\sum |\gamma_i| \leq 1$. Since $v \in L$, $\operatorname{lin}\{e_i\} \subset L$, and $-\frac{1}{2}z \notin L$, it follows that the sum $\sum \gamma_i e_i$ contains infinitely many nonzero members. Therefore, for each positive integer m there is a positive integer n_m such that $\frac{1}{2}\sum_{n_m+1}^{\infty} |\xi_i| < \sum_{m+1}^{\infty} |\gamma_i|$. Let $z_m = -\frac{1}{2}\sum_1^{\infty} \xi_i e_i + \sum_1^m \gamma_i e_i + \frac{1}{2}\sum_{n_m+1}^{\infty} \xi_i e_i$. Then $z_m \in A$, and $z_m = -\frac{1}{2}\sum_1^{n_m} \xi_i e_i + \sum_1^m \gamma_i e_i \in \operatorname{lin}\{e_i\}$. Thus, by (5), $h(z_m) = \hat{f}(z_m)$, but $\lim z_m = v$, so $\hat{f}(v) = \lim \hat{f}(z_m) = \lim h(z_m) = h(v)$. From $\hat{f}/A = h/A$ we easily get

$$\sup \hat{f}(A) = \sup h(A) = \sup h(\operatorname{cl} A). \tag{6}$$

Let $g = T^{-1}x_0 = -\frac{1}{2}z + \sum a_i e_i$ where $\sum |a_i| \leq 1$. Hence $g \in A$, and by (6) and (1) we have

$$\begin{aligned} h(g) &= \hat{f}(g) = f(x_0) = \sup f(W) = \sup \hat{f}(A) \\ &= \sup h(A) = \sup h(\operatorname{cl} A) = \sup\ h(-\tfrac{1}{2}z + U(\ell_1)). \end{aligned}$$

Thus $h(\sum a_i e_i) = \sup h(U(\ell_1))$. Since $h \in c_0$, we see that the sum $\sum a_i e_i$ contains only a finite number (say, m) of nonzero members. So $x_0 = -\frac{1}{2}y + \sum_1^m a_i x_i$, but $x_0 \in W \subset M$ and $\sum_1^m a_i x_i \in M$ while $y \notin M$. This contradiction completes the proof. □

The following corollary is a simple combination of the Theorem with the previously mentioned famous results of Bishop and Phelps.

Corollary 1. *Let X be a separable normed space. The following assertions are equivalent*:

(1) *Every CBC subset $W \subset X$ has a support point.*
(2) *Every CBC subset $W \subset X$ has its support points dense in the boundary of the set W.*
(3) *Every CBC subset $W \subset X$ has a support functional $f \in X^*$.*
(4) *For every CBC subset $W \subset X$ the set of all support functionals $\sum(W)$ of the set W is dense in the dual space X^*.*
(5) *The space X is complete* (*i.e.*, *X is a Banach space*).

The next corollary gives an answer to a question of Klee [1].

Corollary 2. *Every incomplete separable normed space M contains a dense subspace M_1 possessing a closed* (*even in M*) *bounded convex subset W such that each nonzero linear functional f on M_1 with continuous restriction f/W does not attain its supremum on the set W.*

Proof. Let W be a subset constructed as in the theorem, and let $M_1 = \operatorname{lin} W$. Then W and M_1 are as required. □

We conclude the paper with the following open question.

Question. Does every incomplete (separable) normed space M contain a closed bounded convex subset W such that each nonzero linear functional f on M with continuous restriction f/W does not attain its supremum on the set W?

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References

1. V. Klee, *On a question of Bishop and Phelps*, Amer. J. Math. **85** (1963), 95–98.
2. ———, *Extremal structure of convex sets.* II, Math. Z. **69** (1958), 90–104.
3. E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. **67** (1961), 97–98.
4. ———, *The support functionals of a convex set*, Convexity, Proc. Sympos. Pure Math., vol. 7, Amer. Math. Soc., Providence, RI, 1963, pp. 27–35.
5. J. M. Borwein and D. W. Tingley, *On supportless convex sets*, Proc. Amer. Math. Soc. **94** (1985), 471–476.
6. V. P. Fonf, *On absorbing supportless convex subsets in normed spaces*, Studia Math. **104** (1993), 279–284.
7. I. Singer, *Bases in Banach spaces*, Vol. 2, Springer-Verlag, Berlin, 1981.
8. A. I. Markushevich, *On basis* (*in the broad sense of the word*) *for linear spaces*, Dokl. Akad. Nauk SSSR **41** (1943), no. 6, 241–244.
9. M. G. Krein, D. P. Milman, and M. A. Rutman, *On one property of basis in a Banach space*, Notes of Math. Soc., Kharkov **16** (1940), 106–110.

Department of Mathematics, Kharkov Institute of Railway Engineers, Sq. Feujerbach 7, 310050 Kharkov, Ukraine

Current address: Department of Mathematics and Computer Science, Ben-Gurion University of the Negev, Beer-Sheva, Israel