RAMSEY-THEORY AND FORCING EXTENSIONS

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Abstract. In every forcing extension of every model of set theory by a non-trivial (set) forcing there exist a graph $Y$ and a cardinal $\mu$, such that every graph has an edge coloring with $\mu$ colors with no monochromatic induced copy of $Y$.

One of the basic questions in the Ramsey theory of graphs is the following. Given the graphs $X$ and $Y$ and a cardinal $\mu$, is it true that whenever the edges of $X$ are $\mu$-colored, there always exists a monochromatic induced copy of $Y$? We denote the affirmative answer by $X \rightarrow (Y)^2_\mu$, the negative one by $X \nrightarrow (Y)^2_\mu$. The question whether for each $Y$, $\mu$ there exists such an $X$ was answered positively for $Y$, $\mu$ finite in the early seventies. It was natural to assume that the answer is positive in the unrestricted case, too. However, in [2], Hajnal and Komjáth showed that adding a Cohen real produces a counterexample, a graph $Y$ of cardinal $\omega_1$ such that for no graph $X$ in the enlarged model does $X \rightarrow (Y)^2_\mu$ hold. Soon after, S. Shelah showed that the “yes” answer is consistent, too; a class forcing gives a model, where for every pair $Y$, $\mu$ an appropriate $X$ exists.

Here we show that there is nothing special about the Cohen forcing: every (set) forcing adds a counterexample, assuming that it adds something.

For a good exposition and more historical details we refer to [1].

We call a poset nontrivial, if every condition has incompatible extensions.

Theorem. If $(P, \leq)$ is any nontrivial (set) notion of forcing, then, in $V^P$, there exist a graph $Y$ and a cardinal $\mu$ such that $X \nrightarrow (Y)^2_\mu$ holds for every graph $X$.

Fix the poset $(P, \leq)$ for the rest of the paper.

Lemma 1. There is a dense set $D = \{p(\alpha): \alpha < \kappa\} \subseteq P$ for some cardinal $\kappa$ such that $p(\alpha) \nleq p(\beta)$ whenever $\alpha < \beta < \kappa$.

This is Hausdorff's theorem for partially ordered sets. For the sake of completeness we supply a proof.

Proof. Let $\kappa$ be the minimal cardinal such that there is a dense set of cardinal $\kappa$, say $E = \{q(\alpha): \alpha < \kappa\}$. Put $\alpha \in X$ if for no $\beta < \alpha$ does $q(\beta) < q(\alpha)$ hold.

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We claim that \( D = \{ q(\alpha) : \alpha \in X \} \) is dense. As this implies that \( |D| = \kappa \), this clearly suffices. Assume that \( p \in P \) is arbitrary. Let \( \alpha < \kappa \) be the least ordinal such that \( q(\alpha) \leq p \) (exists as \( E \) is dense). \( \alpha \in X \), as otherwise, for some \( \beta < \alpha \), \( q(\beta) < q(\alpha) \leq p \), so \( \alpha \) was not minimal. If now \( X = \{ \xi(\alpha) : \alpha < \kappa \} \) is the increasing enumeration of \( X \), then \( p(\alpha) = q(\xi(\alpha)) \) is as required.

**Lemma 2.** There is a function \( T : [2^\kappa]^2 \rightarrow \kappa \) such that whenever \( h : 2^\kappa \rightarrow \kappa \), there is an \( i < \kappa \) such that \( T \) assumes every value \( < \kappa \) on \( h^{-1}(\{i\}) \).

**Proof.** In the proof we repeatedly use König's theorem, i.e., that \( \text{cf}(2^\kappa) > \kappa \). As \( (2^\kappa)^\kappa = 2^\kappa \) it is possible to define \( T : [2^\kappa]^2 \rightarrow \kappa \) with the following property. If \( x(\alpha) < 2^\kappa \) are different and \( i(\alpha) < \kappa \) for \( \alpha < \kappa \), then there exist arbitrary large \( y < 2^\kappa \) such that \( T(x(\alpha), y) = i(\alpha) \) for all \( \alpha < \kappa \). We show that this \( T \) works.

Assume that \( h : 2^\kappa \rightarrow \kappa \) is a counterexample and that the color class \( h^{-1}(\{i\}) \) misses some \( i(\alpha) < \kappa \). Put \( S = \{ \alpha : \sup(h^{-1}(\{\alpha\})) = 2^\kappa \} \). Select \( z < 2^\kappa \) so large that \( \sup(h^{-1}(\{\alpha\})) < z \) for \( \alpha \notin S \) and \( h^{-1}(\{\alpha\}) \) has at least \( \kappa \) elements below \( z \) for \( \alpha \in S \). Then we can choose different \( x(\alpha) < z \), \( h(x(\alpha)) = \alpha \) (\( \alpha \in S \)) and find \( y > z \) with \( T(x(\alpha), y) = i(\alpha) \) (by the way \( T \) was constructed). If now \( h(y) = \alpha \), clearly \( \alpha \in S \), so then \( h^{-1}(\{i\}) \) does not miss color \( i(\alpha) \), a contradiction.

Given \( \{ p(\alpha) : \alpha < \kappa \} \) as in Lemma 1 and \( T \) as in Lemma 2, assume that \( G \subseteq P \) is a generic set. We construct the graph \( Y \) in the theorem as follows. The vertex set will be \( 2^\kappa \). For \( \alpha < \beta < 2^\kappa \) we let \( \{ \alpha, \beta \} \) be an edge of \( Y \) iff \( p(T(\alpha, \beta)) \in G \). Assume that \( 1 \models X \) is a graph on \( \lambda \). In \( V[G] \), if \( \alpha < \beta < \lambda \) and \( \{ \alpha, \beta \} \in X \), then some element of \( D \cap G \) forces this, as \( D \) is dense, and those conditions determining the truth value of \( \{ \alpha, \beta \} \in X \) form a dense, open set, so \( G \) meets the intersection of these two sets and the condition in the intersection cannot force \( \{ \alpha, \beta \} \notin X \). Color the edge \( \{ \alpha, \beta \} \) by the least \( \xi < \kappa \) such that \( p(\xi) \in G \) forces \( \{ \alpha, \beta \} \in X \).

Assume that there are \( p \in P \), \( i < \kappa \) such that \( p \not\vdash f : 2^\kappa \rightarrow \lambda \) embeds \( Y \) into the \( i \)th color of \( X \). Clearly, \( p \) and \( p(i) \) are compatible (as otherwise \( p \) forces that no edge gets color \( i \)), so we may as well assume that \( p \leq p(i) \). For every \( \alpha < 2^\kappa \) select \( \gamma(\alpha) < \kappa \), \( g(\alpha) < \lambda \) such that \( p(\gamma(\alpha)) \leq p(\gamma(\alpha)) \not\vdash f(\alpha) = g(\alpha) \). By Lemma 2, there is a \( j < \kappa \) such that \( T \) assumes every value \( < \kappa \) in \( \{ \alpha < 2^\kappa : \gamma(\alpha) = j \} \). Select incompatible \( p(\delta) \), \( p(\delta') < p(j) \). As \( p(\delta) < p(j) \leq p \leq p(i) \), by Lemma 1, \( i \leq j < \delta \) holds. There are \( \alpha < \beta < 2^\kappa \) with \( \gamma(\alpha) = \gamma(\beta) = j \), \( T(\alpha, \beta) = \delta \). Then \( p(\delta) \) forces that \( \{ \alpha, \beta \} \in Y \), \( f(\alpha) = g(\alpha) \), \( f(\beta) = g(\beta) \) which imply that \( \{ g(\alpha), g(\beta) \} \in X \). By the way we colored the edges of \( X \), \( p(i) \not\vdash \{ g(\alpha), g(\beta) \} \in X \), which cannot be true as \( p(\delta') \) forces that \( f(\alpha) = g(\alpha) \), \( f(\beta) = g(\beta) \), but it also forces that \( p(\delta) \not\in G \) and so \( \{ \alpha, \beta \} \notin Y \), so \( \{ g(\alpha), g(\beta) \} \notin X \).

**References**


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