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SOLVABILITY OF THE EQUATION $\Delta_g u + \widetilde{S} u^\sigma = Su$ ON MANIFOLDS

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ABSTRACT. For the negative Yamabe invariant and $\widetilde{S} \leq 0$, we obtain that the equation $\Delta_g u + \widetilde{S} u^\sigma = Su$ has a positive solution if and only if the supremum of the Yamabe invariant over all smooth coverings of the 0-level set of $\widetilde{S}$ is positive.

0. INTRODUCTION

Suppose M is a compact manifold of dimension $n \geq 3$ with a Riemannian metric g. The *conformal scalar curvature equation* on M is as follows:

$$(*) \qquad \Delta_g u + \widetilde{S} u^\sigma = Su$$

on M, where $\sigma = (n+2)/(n-2)$, Δ_g is the *Laplacian* of g, $R = 4(n-1)S/(n-2)$ is the scalar curvature of g, and $\widetilde{R} = 4(n-1)\widetilde{S}/(n-2)$ is the prescribed scalar curvature of $\tilde{g} = u^{4/(n-2)} g$. The linear part of $(*)$ is the *conformal Laplacian* of g on M, $L_g = -\Delta_g + S$. Using the conformal Laplacian L_g, we define the *Yamabe invariant* (or *Sobolev quotient*) of g on M as

$$Q(M, g) = \inf_{\phi \in C^\infty(M)} \frac{\langle \phi, L_g \phi \rangle_g}{\|\phi\|_p^2}$$

where $\langle\ ,\ \rangle_g$ is the L^2-inner product with respect to the Riemannian measure of g and $\|\cdot\|_p$ is the L^p-norm with $p = 2n/(n-2)$. The Yamabe invariant is a conformal invariant; i.e., it is a constant in the conformal class of g (cf. [Au]).

In §I we shall consider equation $(*)$ in the case of $Q(M, g) < 0$. In this case, by a conformal deformation of g on M we may arrange S to be a negative constant $-\lambda$ $(\lambda > 0)$ (cf. [Tr]). So, equation $(*)$ can be written as

$$\text{(I)} \qquad \Delta_g u + \widetilde{S} u^\sigma = -\lambda u$$

on M. In [KW1], Kazdan and Warner obtained that, if $\int \widetilde{S}\, dV_g < 0$, then there is a positive constant $\gamma_0 = \gamma_0(\widetilde{S})$ such that for $0 < \lambda < \gamma_0$ equation (I) has a positive solution; for $\lambda > \gamma_0$ it has no positive solutions. Moreover,

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if $\widetilde{S} < 0$, then the constant $\gamma_0 = \infty$; i.e., for all $\lambda > 0$ equation (I) has a positive solution. Kazdan and Warner further conjectured that the constant $\gamma_0 = \infty$ if and only if $\widetilde{S} \leq 0$, which is similar to the result on Riemann surfaces (cf. [KW2]). In §I we shall show that Kazdan and Warner's conjecture is false and give a fairly complete answer to this problem (see Corollary 1 in §I). After obtaining this result we learned that Ouyang has worked on this problem independently, using a different method (cf. [Ou]). We note that our approach is much shorter and that our results differ somewhat from Ouyang's (cf. Remark following Proposition 1 in §I).

From a geometric point of view, it is desirable to obtain a conformally invariant condition on the solvability of (I). In fact, such a conformally invariant condition will then apply to the solvability of $(*)$ as well. In §II we obtain a conformally invariant condition for $Q(M, g) < 0$ and $\widetilde{S} \leq 0$: equation $(*)$ has a positive solution on M if and only if the supremum of the Yamabe invariant over all smooth coverings of the 0-level set of $\widetilde{S}$ is positive (see Theorem A′ in §II).

We also consider in §III the solvability of $(*)$ on the noncompact manifold obtained by deleting a submanifold from M. Let $\Gamma \subset M$ be a closed submanifold of dimension k, and let $\widehat{M} = M \backslash \Gamma$. Denote $\rho(x) = \operatorname{dist}(x, \Gamma)$. In [Mc] McOwen recently posed a problem as follows: suppose $k > (n-2)/2$ and the function $\widetilde{S} \in C^\infty(\widehat{M})$ satisfies

$$-A_1 \leq \widetilde{S} \leq -A_2 \quad \text{as } \rho \to 0,$$

for $A_1 \geq A_2 > 0$ (but $\widetilde{S}$ could be zero or positive somewhere in $\widehat{M}$); does equation $(*)$ have a positive solution u on $\widehat{M}$ satisfying

$$C_1 \rho^{-2/(\sigma-1)} \geq u \geq C_2 \rho^{-2/(\sigma-1)} \quad \text{as } \rho \to 0,$$

where $C_1 > C_2 > 0$? In §III we obtain complete results on the case that $\widetilde{S} \leq 0$ (see Theorem B′ in §III).

I. The case on compact manifolds

In this section, we investigate equation (I) on M. Let us define the first eigenvalue of $-\Delta_g$ on open sets with the Dirichlet boundary condition and the constant λ^* for a function $\widetilde{S}$ which will decide the existence of a positive solution of (I).

Definition 1. (a) For a connected open set U with one-codimensional smooth boundary in M, define the *first eigenvalue* of $-\Delta_g$ on U to be

$$\lambda_1(-\Delta_g; U) = \inf_{\phi \in C_0^\infty(U)} \frac{\int |\nabla \phi|^2 \, dV_g}{\int \phi^2 \, dV_g}.$$

(b) Denote by Π the collection of all open sets V of the form $V = \bigcup\{U_i : i = 1, 2, \dots\}$, where the union is disjoint and each U_i is open with one-codimensional smooth boundary, and define the *first eigenvalue* of $-\Delta_g$ on V to be

$$\lambda_1(-\Delta_g; V) = \inf\{\lambda_1(-\Delta_g; U_i) : i\}.$$

Definition 2. Let a function $\widetilde{S} \in C^\infty(M)$ be negative somewhere, and let Ω_c denote the c-level set of $\widetilde{S}$: $\Omega_c = \{x \in M : \widetilde{S}(x) \geq c\}$. Define the constant λ^*

for the 0-level set Ω_0 of $\widetilde{S}$ by

$$\lambda^* = \sup\{\lambda_1(-\Delta_g\,;\,V)\colon V \supset \Omega_0 \text{ and } V \in \Pi\}.$$

Remarks. In Definition 1(b), the volume of V is always finite. If V has infinitely many components U_i, then $\mathrm{Vol}(U_i) \to 0$, which implies $\lambda_1(-\Delta_g\,;\,U_i) \to \infty$. Therefore, the infimum in (b) can always be reached. In Definition 2, the condition that $\widetilde{S}$ is negative somewhere is reasonable, since $\int \widetilde{S}\,dV_g < 0$ is a necessary condition for the existence of a positive solution of (I) (cf. [KW1, Proposition 4.5]). Also, $\Omega_0 \neq M$ implies that the constant λ^* is positive or infinite.

Next, we introduce the following main theorem.

Theorem A. *Suppose a function $\widetilde{S} \in C^\infty(M)$ is nonpositive on M and negative somewhere.*

(a) *For $0 < \lambda < \lambda^*$ equation* (I) *has a positive solution.*
(b) *If $\lambda^* < \infty$, then for $\lambda \geq \lambda^*$ it has no positive solutions.*

Before proving Theorem A, we first investigate the following lemma and propositions. The lemma is straightforward but gives a more concrete description of the constant λ^* in terms of the limit of the first eigenvalues on the level sets of $\widetilde{S}$.

Lemma 1. *Suppose a function $\widetilde{S} \in C^\infty(M)$ is negative somewhere and its 0-level set $\Omega_0 \neq \varnothing$. Then there is a strictly decreasing positive sequence $\{\varepsilon_i\}$ with $\varepsilon_i \to 0$ such that $\Omega_{-\varepsilon_{i+1}} \subset \overset{\circ}{\Omega}_{-\varepsilon_i}$ and $\overset{\circ}{\Omega}_{-\varepsilon_i} \in \Pi$ for each i, where $\overset{\circ}{\Omega}_{-\varepsilon_i}$ is the interior of $\Omega_{-\varepsilon_i}$. Moreover, $\Omega_0 = \bigcap_i \Omega_{-\varepsilon_i}$, and the sequence $\{\lambda_1(-\Delta_g\,;\,\overset{\circ}{\Omega}_{-\varepsilon_i})\}$ of the first eigenvalues is strictly increasing and has the constant λ^* as its limit*

$$\lambda^* = \lim_{i\to\infty} \lambda_1(-\Delta_g\,;\,\overset{\circ}{\Omega}_{-\varepsilon_i}).$$

Proof. By Sard's Theorem, let $\{\varepsilon_i\}$ be a strictly decreasing positive sequence of the regular values of $\widetilde{S}$ with $\varepsilon_i \to 0$. From the Implicit Function Theorem, all the $\overset{\circ}{\Omega}_{-\varepsilon_i}$ are in Π. Thus, this lemma follows. □

In the following proposition, we observe that any closed set can be the 0-level set of some nonpositive function in $C^\infty(M)$.

Proposition 1. *For any closed set C on M, $C \neq M$, there exists a nonpositive function $\widetilde{S} \in C^\infty(M)$ such that $\widetilde{S} < 0$ on $M\backslash C$, and its 0-level set $\Omega_0 = C$.*

Proof. Note that $M\backslash C$ is a nonempty open set. By the analysis on manifolds, we can have a sequence $\{U_i\}$ of small open sets with $M\backslash C = \bigcup U_i$ such that, for each i, there is a function $\phi_i \in C^\infty(M)$ satisfying

$$\phi_i < 0 \quad \text{on } U_i, \qquad \phi_i = 0 \quad \text{on } M\backslash U_i.$$

Let $M_{i,k} = |\phi_i|_{C^k(M)}$. Then $0 < M_{i,k} \leq M_{i,k+1}$ for all k. Define a function $\widetilde{S}$ to be

$$\widetilde{S} = \sum \frac{\phi_i}{2^i M_{i,i}} \quad \text{on } M.$$

So $\widetilde{S}$ is well defined in $C^\infty(M)$ and meets the requirements. □

Remark. From Lemma 1, the sequence $\{\Omega_{-\varepsilon_i}\}$ decreases to Ω_0, and the constant λ^* is the limit of the first eigenvalues on $\mathring{\Omega}_{-\varepsilon_i}$. It is well known that the eigenvalues vary continuously under continuous deformations of domains (cf. [CH]). But if the deformations of the domains are irregular, i.e., the topological type is not preserved or some part of the domain degenerates, the analysis of convergence of the eigenvalues is much more complicated and is a very important topic. Thus, when $\Omega_{-\varepsilon}$ is a continuous deformation of Ω_0, the sequence $\{\lambda_1(-\Delta_g\,;\mathring{\Omega}_{-\varepsilon_i})\}$ converges to $\lambda_1(-\Delta_g\,;\mathring{\Omega}_0)$, where $\mathring{\Omega}_0$ is the interior of Ω_0. In this case, the constant $\lambda^* = \lambda_1(-\Delta_g\,;\mathring{\Omega}_0)$ as in [Ou]. On the other hand, if $\Omega_{-\varepsilon}$ is an irregular deformation of Ω_0, it may be that the constant λ^* is not equal to $\lambda_1(-\Delta_g\,;\mathring{\Omega}_0)$ (cf. Example 2).

Without assuming that $\widetilde{S}$ is nonpositive, we observe that for $\lambda \geq \lambda^*$ there are no positive solutions to equation (I) in the more general case.

Proposition 2. *Suppose a function $\widetilde{S} \in C^\infty(M)$ is negative somewhere. If $\lambda^* < \infty$, then, for $\lambda \geq \lambda^*$, equation* (I) *has no positive solutions.*

Proof. Assume that equation (I) has a positive solution $u \in C^\infty(M)$ for $\lambda \geq \lambda^*$. Set $a = \min\{u(x)\colon x \in M\} > 0$. Let the sequence $\{\Omega_{-\varepsilon_i}\}$ of the level sets be obtained in Lemma 1. Thus, we may assume that $\lambda^*/2 < \lambda_1(-\Delta_g\,;\mathring{\Omega}_{-\varepsilon_i}) < \lambda^*$. Since $\lambda \geq \lambda^* > \lambda_1(-\Delta_g\,;\mathring{\Omega}_{-\varepsilon_i})$, there is a connected open component $U_{-\varepsilon_i}$ of $\mathring{\Omega}_{-\varepsilon_i}$ such that $\lambda_1(-\Delta_g\,;U_{-\varepsilon_i}) < \lambda$. Let ϕ be the eigenfunction corresponding to the first eigenvalue $\lambda_1(-\Delta_g\,;U_{-\varepsilon_i})$:

$$\begin{aligned} (-\Delta_g - \lambda)\phi &= -\overline{\delta}\phi \quad \text{in } U_{-\varepsilon_i}, \\ \phi &> 0 \quad \text{in } U_{-\varepsilon_i}, \\ \phi &= 0 \quad \text{on } \partial U_{-\varepsilon_i} \end{aligned}$$

where $\overline{\delta} = \lambda - \lambda_1(-\Delta_g\,;U_{-\varepsilon_i}) > 0$. Using Green's Identity, we have

$$\begin{aligned} \int_{\partial U_{-\varepsilon_i}} \left(u\frac{\partial\phi}{\partial n} - \phi\frac{\partial u}{\partial n}\right) dS_g &= \int_{U_{-\varepsilon_i}} (u\Delta_g\phi - \phi\Delta_g u)\, dV_g \\ &= \int_{U_{-\varepsilon_i}} (\overline{\delta} + \widetilde{S}u^{\sigma-1})u\phi\, dV_g > -\varepsilon_i \int_{U_{-\varepsilon_i}} u^\sigma\phi\, dV_g \end{aligned}$$

where n is the exterior unit normal and dS_g is the induced measure of g on $\partial U_{-\varepsilon_i}$. Note that $\partial\phi/\partial n \leq 0$ on $\partial U_{-\varepsilon_i}$. We have

$$\begin{aligned} \int_{\partial U_{-\varepsilon_i}} \left(u\frac{\partial\phi}{\partial n} - \phi\frac{\partial u}{\partial n}\right) dS_g &\leq a\int_{\partial U_{-\varepsilon_i}} \frac{\partial\phi}{\partial n}\, dS_g \\ &= -a\lambda_1(-\Delta_g\,;U_{-\varepsilon_i})\int_{U_{-\varepsilon_i}} \phi\, dV_g < -\frac{a\lambda^*}{2}\int_{U_{-\varepsilon_i}} \phi\, dV_g, \end{aligned}$$

so

$$\int_{U_{-\varepsilon_i}} \left(u^\sigma\varepsilon_i - \frac{a\lambda^*}{2}\right)\phi\, dV_g > 0.$$

As $\varepsilon_i \to 0$, this is a contradiction. □

We shall prove Theorem A as follows:

Proof of Theorem A. To prove part (a) of Theorem A, we construct a positive upper solution $u_+ \in C^\infty(M)$ to equation (I) for $0 < \lambda < \lambda^*$, i.e.,

$$\Delta_g u_+ + \widetilde{S}u_+^\sigma \leq -\lambda u_+ \quad \text{on } M.$$

By [KW1], there exists a positive lower solution $u_- \in C^\infty(M)$ to equation (I) for $\lambda > 0$, i.e., $u_- \leq u_+$, and

$$\Delta_g u_- + \widetilde{S}u_-^\sigma \geq -\lambda u_- \quad \text{on } M.$$

So, we can obtain a positive solution to equation (I) for $0 < \lambda < \lambda^*$ (see [KW1, Lemmas 2.6 and 2.8]). For $0 < \lambda < \lambda^*$, there is an open set $V \in \Pi$ such that $V \supset \Omega_0$ and $\lambda_1(-\Delta_g; V) > \lambda$. Write $V = \bigcup U_i$, a union of disjoint connected open components. Since Ω_0 is compact, we may assume that $V = U_1 \cup \cdots \cup U_m$. Note that $\lambda_1(-\Delta_g; U_k) > \lambda$ for each k. Let ϕ_k be the eigenfunction corresponding to the first eigenvalue $\lambda_1(-\Delta_g; U_k)$:

$$\begin{aligned} (-\Delta_g - \lambda)\phi_k &= \varepsilon_k \phi_k && \text{in } U_k, \\ \phi_k &> 0 && \text{in } U_k, \\ \phi_k &= 0 && \text{on } \partial U_k, \end{aligned}$$

$\varepsilon_k = \lambda_1(-\Delta_g; U_k) - \lambda > 0$. For each U_k, we can choose a smaller connected open set U_k' in Π such that $U_k' \cup \partial U_k' \subset U_k$ and $V' = \bigcup U_i' \supset \Omega_0$. So

$$(-\Delta_g - \lambda)\phi_k = \varepsilon_k \phi_k \quad \text{on } U_k' \cup \partial U_k'$$

and $\phi_k > 0$ on $U_k' \cup \partial U_k'$. Let $w \in C^\infty(M)$ be a positive function satisfying

$$w = \phi_k \quad \text{on } U_k' \cup \partial U_k'$$

for each k. Set $u_+ = cw$, where c is a positive constant. Then, for any $c > 0$,

$$\Delta_g u_+ + \widetilde{S}u_+^\sigma \leq -\lambda u_+ \quad \text{on } U_k' \cup \partial U_k'.$$

Since $\max\{\widetilde{S}w^\sigma : x \in M \backslash V'\} \leq -\varepsilon$ for some $\varepsilon > 0$, and $-\Delta_g w - \lambda w$ is bounded on M, letting c be large enough, we have

$$c^{\sigma-1}\widetilde{S}w^\sigma \leq -\Delta_g w - \lambda w \quad \text{on } M \backslash V',$$

which implies

$$\Delta_g u_+ + \widetilde{S}u_+^\sigma \leq -\lambda u_+ \quad \text{on } M \backslash V'.$$

So u_+ is a positive upper solution of (I). Part (b) directly follows from Proposition 2. □

Finally, using Theorem A, we shall give an answer to the original Kazdan and Warner problem.

Corollary 1. *Equation* (I) *has a positive solution for all* $\lambda > 0$ *if and only if the function* $\widetilde{S}$ *is nonpositive on* M *and negative somewhere and the constant* λ^* *for* $\widetilde{S}$ *is infinite.*

Proof. Suppose equation (I) has a positive solution for all $\lambda > 0$. By Proposition 4.5 and 4.10 in [KW1], $\widetilde{S} \leq 0$ on M and $\widetilde{S}$ is negative somewhere.

From part (b) of Theorem A, it follows that $\lambda^* = \infty$. Conversely, by part (a) of Theorem A, equation (I) has a positive solution for all $\lambda > 0$. □

Corollary 1 applies to many functions $\widetilde{S}$, for example, any for which Ω_0 has measure zero. However, it may even apply to Ω_0 having positive measure as the following Example 1 shows. We also give Example 2 in which the interior of Ω_0 is empty and $\lambda^* < \infty$; this gives a counterexample to Theorem 1 in [Ou].

Example 1. Suppose $\widetilde{S} \in C^\infty(M)$ with $\widetilde{S} \leq 0$ and the 0-level set Ω_0 being a *Cantor set* C in M, i.e., $\mathrm{Vol}(C) > 0$ but each connected component of C has zero volume. (Notice that, given such a Cantor set C in M, we can use Proposition 1 to construct such an $\widetilde{S}$ with $\Omega_0 = C$.) By Corollary 1, it suffices to prove that $\lambda^* = \infty$. Note that, for any $\varepsilon > 0$, there exists an open set $V \in \Pi$ with $V = \bigcup U_i$, a union of disjoint connected components, such that $V \supset \Omega_0$, and $\mathrm{Vol}(U_i) < \varepsilon$ for each i. So, for any $b > 0$, by the lower bound of the first eigenvalue (cf. [Li]), there exists an open set $V \in \Pi$ such that $V \supset \Omega_0$ and $\lambda_1(-\Delta_g; V) \geq b$, which shows that $\lambda^* = \infty$.

Example 2. Let $U \in \Pi$ be a connected domain in M, and let $\Lambda = \{x_1, x_2, \dots\}$ be a countable set in U and dense in U. Denote $B_\varepsilon(x) = \{y \in M\colon \mathrm{dist}(y, x) < \varepsilon\}$. First, consider the domain $U_1 = U - \overline{B_{\varepsilon_1}(x_1)} \subset U$ as $\varepsilon_1 \to 0$. Let $\lambda_1 = \lambda_1(-\Delta_g; U)$ and $\lambda(\varepsilon_1) = \lambda_1(-\Delta_g; U_1)$. By Proposition 1 in [Oz], $\lambda(\varepsilon_1) \to \lambda_1$ as $\varepsilon_1 \to 0$. Thus, we are able to choose ε_1 small enough so that $\lambda(\varepsilon_1) \leq \lambda_1 + \frac{1}{2}$. Second, if $x_2 \notin U_1$, take it out and check x_3. Otherwise, consider the domain $U_2 = U_1 - \overline{B_{\varepsilon_2}(x_2)} \subset U_1$ as $\varepsilon_2 \to 0$. Let $\lambda(\varepsilon_2) = \lambda_1(-\Delta_g; U_2)$. Since $\lambda(\varepsilon_2) \to \lambda(\varepsilon_1)$ as $\varepsilon_2 \to 0$, we can have a sufficiently small ε_2 such that $\lambda(\varepsilon_2) \leq \lambda_1 + \frac{1}{2} + \frac{1}{4}$. Then we keep continuing these processes and obtain a sequence $\{U_i\}$ of the domains such that $U_{i+1} \subset U_i$ and $\lambda(\varepsilon_i) \leq \lambda_1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^i} < \lambda_1 + 1$ for all i. By Proposition 1, we construct a nonpositive function $\widetilde{S} \in C^\infty(M)$ such that its 0-level set $\Omega_0 = \bigcap \overline{U}_i$. For any open set $V \in \Pi$ and $V \supset \Omega_0$, there is some domain U_k with $U_k \subset V$. Thus, $\lambda_1(-\Delta_g; V) < \lambda(\varepsilon_k) < \lambda_1 + 1$, which implies the constant $\lambda^* < \infty$. On the other hand, the interior of Ω_0 is clearly empty. Thus the parameter $\overline{\lambda}$ of [Ou] is $\overline{\lambda} = +\infty$, and we see that Theorem A contradicts Theorem 1 in [Ou].

II. A CONFORMALLY INVARIANT CONDITION

In this section, we shall obtain a conformally invariant condition on the solvability of equation $(*)$. Suppose $Q(M, g) < 0$ and $\widetilde{S} \in C^\infty(M)$ satisfies $\widetilde{S} \leq 0$ in M and $\widetilde{S} < 0$ somewhere. Then there exists $g_1 = u_1^{4/(n-2)} g$ with a negative scalar curvature $-4(n-1)\lambda/(n-2)$. Thus, we may change equation $(*)$ into

$$\text{(I}'\text{)} \qquad \Delta_{g_1} v + \widetilde{S} v^\sigma = -\lambda v,$$

and let the solution of $(*)$ be $u = v u_1$.

Using the conformal Laplacian $L_{g_1} = -\Delta_{g_1} - \lambda$ of g_1, we give the following definition.

Definition 3. (a) For a connected open set U with one-codimensional smooth boundary in M, define the *first eigenvalue* of L_{g_1} and the *Yamabe invariant* of

g_1 on U, respectively, to be

$$\lambda_1(L_{g_1}; U) = \inf_{\phi \in C_0^\infty(U)} \frac{\langle \phi, L_{g_1}\phi\rangle_{g_1}}{\|\phi\|_2^2},$$

$$Q(U, g_1) = \inf_{\phi \in C_0^\infty(U)} \frac{\langle \phi, L_{g_1}\phi\rangle_{g_1}}{\|\phi\|_p^2} = \inf_{\substack{\phi \in C_0^\infty(U) \\ \|\phi\|_p = 1}} \|\nabla\phi\|_2^2 - \lambda\|\phi\|_2^2$$

where $\langle\ ,\ \rangle_{g_1}$ is the L^2-inner product with respect to the Riemannian measure of g_1 and $\|\cdot\|_p$ is the L^p-norm with $p = 2n/(n-2)$.

(b) Denote by Π' the collection of all open sets V of the form $V = \bigcup U_i$, where the union is disjoint and finite, each U_i is open with one-codimensional smooth boundary, and define the *first eigenvalue* of L_{g_1} and the *Yamabe invariant* of g_1 on V, respectively, to be

$$\lambda_1(L_{g_1}; V) = \inf\{\lambda_1(L_{g_1}; U_i) : i\},$$
$$Q(V, g_1) = \inf\{Q(U_i, g_1) : i\}.$$

Remark. It is well known that the Yamabe invariant is conformally invariant on M (cf. [Au]). For $Q(V, g_1)$, we can use $\phi \in C_0^\infty(V)$ to prove easily that $Q(V, g_1)$ is also conformally invariant:

$$Q(V, g_1) = Q(V, g). \tag{1}$$

Since the 0-level set Ω_0 of $\widetilde{S}$ is compact, we can replace Π by Π' in Definition 2. Therefore, we have

$$\lambda^* = \sup\{\lambda_1(-\Delta_{g_1}; V) : V \supset \Omega_0 \text{ and } V \in \Pi'\} \tag{2}$$

and the following corollary.

Corollary 2. *Equation* (I′) *has a positive solution if and only if*

$$\sup\{\lambda_1(L_{g_1}; V) : V \supset \Omega_0 \textit{ and } V \in \Pi'\} > 0. \tag{3}$$

Proof. By Theorem A, equation (I′) has a positive solution if and only if $\lambda < \lambda^*$, which implies condition (3). □

To get a conformally invariant condition, we need a lemma as follows:

Lemma 2. *For a connected open set U with one-codimensional smooth boundary in M, $\lambda_1(L_{g_1}; U) > 0$ if and only if $Q(U, g_1) > 0$.*

Proof. If $Q(U, g_1) > 0$, by the Hölder inequality, we have

$$Q(U, g_1) \le c\lambda_1(L_{g_1}; U)$$

where the constant $c = c(n, U) > 0$. So $Q(U, g_1) > 0$ implies $\lambda_1(L_{g_1}; U) > 0$. Now we show that, if $\lambda_1(L_{g_1}; U) > 0$, then $Q(U, g_1) > 0$. Assume that $Q(U, g_1) = 0$. So there is a sequence of $\phi_i \in C_0^\infty(U)$ with $\|\phi_i\|_p = 1$ such that

$$\|\nabla\phi_i\|_2^2 - \lambda\|\phi_i\|_2^2 \to 0 \quad \text{as } i \to \infty. \tag{4}$$

Since $\lambda_1(L_{g_1}; U) > 0$, there is $\varepsilon > 0$ such that

$$\|\nabla\phi_i\|_2^2 - \lambda\|\phi_i\|_2^2 \ge \varepsilon\|\phi_i\|_2^2 \quad \text{for all } i. \tag{5}$$

It follows, from the Hölder inequality and $\|\phi_i\|_p = 1$, that $\|\phi_i\|_2^2$ is bounded. Thus, by (4) and (5), we must have

$$\|\phi_i\|_2 \to 0 \quad \text{and} \quad \|\nabla\phi_i\|_2 \to 0 \quad \text{as } i \to \infty. \tag{6}$$

By the Sobolev inequality, for $\phi_i \in C_0^\infty(U)$,

$$\|\phi_i\|_p \le C\|\nabla\phi_i\|_2$$

where the constant $C = C(n, p) > 0$ (cf. [Li]). Thus, by (6), we have $\|\phi_i\|_p \to 0$ as $i \to \infty$, contradicting $\|\phi_i\|_p = 1$. □

Using Corollary 2 and Lemma 2, we give a conformally invariant condition on the solvability of $(*)$.

Theorem A′. *Suppose $Q(M, g) < 0$ and $\widetilde{S} \in C^\infty(M)$ satisfies $\widetilde{S} \le 0$ in M and $\widetilde{S} < 0$ somewhere. Equation $(*)$ has a positive solution on M if and only if*

$$\sup\{Q(V, g) \colon V \supset \Omega_0 \text{ and } V \in \Pi'\} > 0. \tag{7}$$

Proof. Note that equation $(*)$ has a positive solution if and only if condition (3) holds. From Lemma 2, condition (3) is equivalent to

$$\sup\{Q(V, g_1) \colon V \supset \Omega_0 \text{ and } V \in \Pi'\} > 0. \tag{8}$$

From (1), condition (7) is necessary and sufficient.

III. The case on noncompact manifolds

In this section, we shall investigate equation $(*)$ on $\widehat{M} = M\backslash\Gamma$, where Γ is a closed smooth submanifold of dimension k and $k > (n-2)/2$. First, we consider equation

$$\Delta_g u - u^\sigma = Su \tag{+}$$

on $\widehat{M}$. By the results in [AM, De, and Ta], equation $(+)$ has two different types positive solutions u_1 and u_2 such that:

(i) for $k > (n-2)/2$, $C_1\rho^{-2/(\sigma-1)} \ge u_1 \ge C_2\rho^{-2/(\sigma-1)}$ as $\rho \to 0$,
(ii) for $(n-2)/2 < k < n-2$, $D_1\rho^{2+k-n} \ge u_2 \ge D_2\rho^{2+k-n}$ as $\rho \to 0$,

where $C_1 > C_2 > 0$ and $D_1 > D_2 > 0$. Now let the metric $g_i = u_i^{4/(n-2)}g$ on $\widehat{M}$, where $i = 1$ or 2, and consider equation

$$\Delta_{g_i} w + \widetilde{S}w^\sigma = -w \quad \text{on } \widehat{M}, \tag{II}$$

where the solution w satisfies

(iii) $B_1 \ge w \ge B_2$ as $\rho \to 0$

for $B_1 > B_2 > 0$. In order to find a positive solution u of $(*)$ satisfying asymptotic behaviour (i) or (ii), we have to find a positive solution w of (II) satisfying (iii). Then let $u = wu_i$. If a solution u of $(*)$ satisfies (i), the metric $\tilde{g} = u^{4/(n-2)}g$ is complete on $\widehat{M}$; if u satisfies (ii), $\tilde{g}$ is not complete on $\widehat{M}$.

Definition 4. Suppose a function $\widetilde{S} \in C^\infty(\widehat{M})$ satisfies

$$-A_1 \leq \widetilde{S} \leq -A_2 \quad \text{as } \rho \to 0,$$

for $A_1 \geq A_2 > 0$. Define the constant μ_i^* for the 0-level set Ω_0 of $\widetilde{S}$ under the metric g_i by

$$\mu_i^* = \sup\{\lambda_1(-\Delta_{g_i}; V) \colon \Omega_0 \subset V,\ V \cup \partial V \subset \widehat{M} \text{ and } V \in \Pi\}.$$

Remark. The metric g_i is singular on Γ, but the hypothesis on $\widetilde{S}$ ensures that Ω_0 is compact in $\widehat{M}$, and hence $\lambda_1(-\Delta_{g_i}; V)$ is finite for some $V \supset \Omega_0$.

Using the same process as in Theorem A, we can obtain the following theorem.

Theorem B. *Suppose a function $\widetilde{S} \in C^\infty(\widehat{M})$ is nonpositive and satisfies*

$$-A_1 \leq \widetilde{S} \leq -A_2 \quad \textit{as } \rho \to 0,$$

for $A_1 \geq A_2 > 0$. Then,

(a) *for $\mu_i^* \leq 1$ equation* (II) *has no positive solutions satisfying* (iii), *and*
(b) *for $\mu_i^* > 1$ equation* (II) *has a positive solution w satisfying* (iii).

By analogy to Theorem A′, we can write Theorem B in a conformally invariant form.

Theorem B′. *Suppose a function $\widetilde{S} \in C^\infty(\widehat{M})$ is nonpositive and satisfies*

$$-A_1 \leq \widetilde{S} \leq -A_2 \quad \textit{as } \rho \to 0,$$

for $A_1 \geq A_2 > 0$.

(a) *When $Q(M, g) \geq 0$, equation* $(*)$ *has a positive solution satisfying* (i) *or* (ii).

(b) *When $Q(M, g) < 0$, equation* $(*)$ *has a positive solution satisfying* (i) *or* (ii) *if and only if*

$$\sup\{Q(V, g) \colon \Omega_0 \subset V,\ V \cup \partial V \subset \widehat{M} \textit{ and } V \in \Pi'\} > 0.$$

Proof. By Theorem B and the proof of Theorem A′, equation $(*)$ has a positive solution satisfying (i) or (ii) if and only if

$$\sup\{Q(V, g) \colon \Omega_0 \subset V,\ V \cup \partial V \subset \widehat{M} \text{ and } V \in \Pi'\} > 0,$$

which implies (b). When $Q(M, g) \geq 0$, this condition certainly holds, which shows (a).

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