A HOMOGENEOUS, GLOBALLY SOLVABLE
DIFFERENTIAL OPERATOR ON A NILPOTENT LIE GROUP
WHICH HAS NO TEMPERED FUNDAMENTAL SOLUTION

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Dedicated to the memory of Lawrence Corwin

Abstract. We present an example of a homogeneous, left-invariant differential
operator on the Heisenberg group \( H_3 \) which admits fundamental solutions but
no tempered ones. This answers a question raised by Corwin in the negative.

Assume that \( N \) is a connected, simply connected nilpotent Lie group with Lie
algebra \( \mathfrak{n} \), and let \( L \in \mathfrak{U}(\mathfrak{n}) \) be a left-invariant differential operator on \( N \). If
\( N \) is abelian, any such \( L \) can be considered as a constant coefficient differential
operator on some Euclidean space \( \mathbb{R}^n \) and, by the theorem of Malgrange and
Ehrenpreis (see [H2]), has a fundamental solution \( F \in \mathcal{D}'(\mathbb{R}^n) \), i.e., \( LF = \delta \),
where \( \delta \) denotes the point measure at the identity. In fact, it was proved later
by Hörmander [H1] and Lojasiewicz [L] that one can even find a tempered
fundamental solution \( F \in \mathcal{S}'(\mathbb{R}^n) \).

The situation becomes drastically different if \( N \) is nonabelian, since then
there exist many operators in \( \mathfrak{U}(\mathfrak{n}) \) which are not even locally solvable. Assume
in the sequel that \( N \) admits a one-parameter family \( \{ \delta_r \}_{r > 0} \) of automorphic
dilations (see [FS]) and that \( L \) is homogeneous, i.e., \( L(\phi \circ \delta_r) = r^m(L\phi) \circ \delta_r \), for
some \( m > 0 \) and every \( \phi \in \mathcal{D}(N) \), \( r > 0 \). Then it is at least true that various
notions of solvability coincide for \( L \). For instance, \( L \) is locally solvable at
some point of \( N \) if and only if \( LC^\infty(N) = C^\infty(N) \), if and only if \( L \) has a
fundamental solution \( F \in \mathcal{S}'(N) \) (see, e.g., [B, M1]). Moreover, if \( L^T \), the
transpose of \( L \), is hypoelliptic, then the same is true of the operator \( LL^T \),
as can be seen by Helffer-Nourrigat’s theorem [HN], and one can make use of
the homogeneity of \( L \) in order to prove that \( LL^T \), hence also \( L \), has even a
tempered fundamental solution [F, G].

So, a natural question, which seems to have been open hitherto, is whether
any solvable, homogeneous, left-invariant differential operator or, more gener-
ally, every globally solvable left-invariant differential operator on a nilpotent Lie
group has a tempered fundamental solution. In the latter form, this question
was raised by Corwin in [C].

The purpose of this note is to present an example on the 7-dimensional...
Heisenberg group $H_3$ which answers this question in the negative, even for homogeneous operators.

Namely, if $X_1, X_2, X_3, Y_1, Y_2, Y_3, U$ denotes the standard basis of the Lie algebra $\mathfrak{h}_3$ of $H_3$ (with nontrivial brackets $[X_j, Y_j] = U$, $j = 1, 2, 3$), we set

$$L = (X_2^2 + Y_1^2) - \lambda (X_2^2 + Y_2^2) + Y_3^2, \quad \lambda \in \mathbb{R}\{0\}.$$  

Adopting the notation used throughout [MR1, MR2, M2], we have $L = \Delta_S$, where $S \in \mathfrak{sp}(3, \mathbb{R})$ is given by the matrix

$$S = \begin{pmatrix} H & 0 \\ 0 & N \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \lambda \\ 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with respect to the basis $X_1, X_2, Y_1, Y_2, X_3, Y_3$. Since $S$ is not semisimple, $L$ is locally solvable for every $\lambda \in \mathbb{R}$ by [MR2, Theorem (i.3)].

Now assume there is some $F \in \mathcal{S}'(H_3)$ such that $LF = \delta$. If $(z, u)$, with $z \in \mathbb{R}^6$, $u \in \mathbb{R}$, denote the usual coordinates of $H_3$, we define as in [MR1]

$$F_u(z, \mu) := f^\mu(z) := \int_{\mathbb{R}} f(z, u) e^{-2\pi i \mu u} du, \quad \mu \in \mathbb{R},$$

for $f \in \mathcal{S}(H_3)$. This partial Fourier transform turns $L$ into the “twisted” differential operator $L^\mu$ given by the formula

$$L^\mu f(z) := (L f)^\mu,$$

if $f \in \mathcal{S}(H_3)$. Let $\delta_r(z, u) = (rz, r^2u)$ denote the usual dilations on $H_3$, and fix a real function $\chi \in C_0^\infty(\mathbb{R}^+)$ with support contained in the interval $[1, 2]$ and $\int \chi(r) dr = 1$. For $\phi \in \mathcal{S}(\mathbb{R}^6)$ and $j = 0, 1$, we set

$$A_j \phi(z, u) := \int_{0}^{\infty} \phi(r^{1/2}z) e^{-2\pi iru} \chi(r) r^j dr.$$

Then $A_j$ defines a continuous linear operator from $\mathcal{S}(\mathbb{R}^6)$ into $\mathcal{S}(H_3)$, a fact which follows easily from the formula

$$A_j \phi = F_u(E_j \phi),$$

where $E_j \phi$ is defined by

$$E_j \phi(z, \mu) := \phi(\mu^{1/2}z) \chi(\mu) \mu^j.$$

Moreover, from (3) one easily sees that

$$L F_u(E_0 \phi) = F_u(E_1(L^T \phi));$$

hence

$$L A_0 \phi = A_1(L^T \phi),$$

where we have set $L := L^1$.  

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Let $A^T \in \mathcal{S}'( \mathbb{R}^6 )$ denote the adjoint operator to $A_1$, and let $K \in \mathcal{S}'( \mathbb{R}^6 )$ be given by $K = A^T(F)$. Then, by (4),
\[
\langle \hat{L}K, \varphi \rangle = \langle K, \hat{L}^T \varphi \rangle = \langle F, A_1 \hat{L}^T \varphi \rangle = \langle F, L \varphi \rangle
\]
\[
= \langle LF, A \varphi \rangle = (A_0 \varphi)(0),
\]
since $L = L^T$. Moreover, since
\[
(A_0 \varphi)(0) = \varphi(0) \int_0^\infty \chi(r) \, dr = \varphi(0),
\]
we see that $\hat{L}K = \delta$, i.e., we have proved the following

**Lemma 1.** Assume $L$ (given by (1)) has a tempered fundamental solution. Then the same is true of $\hat{L}$.

Finally, we can invoke [M2] in order to prove

**Proposition 2.** Let $L$ be given by (1). Then:

(i) $L$ has a fundamental solution $F \in \mathcal{S}'( \mathbb{H})$ for every $\lambda \in \mathbb{R}$.

(ii) If $L$ has a tempered fundamental solution $F \in \mathcal{S}'( \mathbb{H})$, then there are constants $C > 0$, $r \in \mathbb{N}$ such that
\[
|\lambda - p/q| > Cq^{-r}
\]
whenever $p$ and $q$ are odd positive integers such that $\lambda - p/q > 0$. In particular, $L$ has no tempered fundamental solution, if $\lambda = \lambda_0$, where $\lambda_0 := \sum_{k=0}^\infty 3^{-k!}$.

**Proof.** It has been shown in [MR2, Proposition 3.9] that the Liouville number $\lambda_0$ violates condition (5), so there remains only to prove (ii).

But, in the notation of [M2], the matrix $S$ associated to $L$ is of type $(E1)$, and condition (5) is equivalent to [M2, Theorem 1.1, condition (1.8)]. Therefore, by [M2, Corollary 3.2 and Theorem 1.1], $L$ can have a tempered fundamental solution only if (5) holds; hence (ii) follows from Lemma 1. Q.E.D.

**Remark 3.** In [M1] we showed that a homogeneous operator $L \in \mathfrak{U}(\mathbb{N})$ is not locally solvable if there is a sequence $\{\psi_j\}_j \subset \mathcal{S}(\mathbb{N})$ such that $\psi_j(0) = 1$ for every $j$ and
\[
\lim_{j \to \infty} \|\psi_j\|_{(\mathbb{N})} \|L^T \psi_j\|_{(\mathbb{N})} = 0
\]
for every Schwartz-norm $\|\cdot\|_{(\mathbb{N})}$. This condition relaxes the necessary condition for local solvability in [CR] and was crucial in [MR2] but may look somewhat unnatural. One is tempted to ask if (6) could be replaced by
\[
\lim_{j \to \infty} \|L^T \psi_j\|_{(\mathbb{N})} = 0.
\]
However, Proposition 2 implies that this is not possible, for, if we could replace (6) by (7), then local solvability of $L$ would imply an estimate of the form
\[
|\psi(0)| \leq \|L^T \psi\|_{(\mathbb{N})}, \quad \psi \in \mathcal{S}(\mathbb{N}),
\]
for some Schwartz-norm $\|\cdot\|_{(\mathbb{N})}$. And, by the Hahn-Banach theorem, this would mean that $L$ had a tempered fundamental solution.

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