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THE CONJUGATION OPERATOR ON $A_q(G)$

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ABSTRACT. Let G be a compact abelian group and Γ its dual. For $1 \le q < \infty$, the space $A_q(G)$ is defined as

$$A_q(G) = \{f | f \in L^1(G), \ \hat{f} \in l_q(\Gamma)\}$$

with the norm $\|f\|_{A_q} = \|f\|_{L^1} + \|\hat{f}\|_{l_q}$. We prove: Let G be a compact, connected abelian group and P any fixed order on Γ. If $q > 2$ and ϕ is a Young's function, then the conjugation operator H does not extend to a bounded operator from $A_q(G)$ to the Orlicz space $L^\phi(G)$.

Let G be a compact abelian group and Γ its dual. For $1 \le q < \infty$, the space $A_q(G)$ is defined as

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with the norm $\|f\|_{A_q} = \|f\|_{L^1} + \|\hat{f}\|_{l_q}$. Then $A_q(G)$ is a commutative semisimple Banach algebra with maximal ideal space Γ, in which the set of trigonometric polynomials $\mathscr{T}$ is dense [4].

If G is, in addition, a connected group, then its dual can be ordered; i.e., there exists a semigroup $P \subseteq \Gamma$ such that $P \cap -P = \{0\}$, $P \cup -P = \Gamma$ [5], and we say that $\gamma \in \Gamma$ is positive if $\gamma \in P$. If $f = \sum_{\gamma \in F} \hat{f}(\gamma)\gamma$ is a trigonometric polynomial, the conjugation operator is defined as

$$Hf = \sum_{\gamma \in F} -i \operatorname{sgn}(\gamma) \hat{f}(\gamma)\gamma$$

where $\operatorname{sgn}(\gamma) = +1$ if $\gamma \in P$, -1 if $\gamma \in -P$, and 0 if $\gamma = 0$.

If $1 \le q \le 2$, then $A_q(G) \subseteq L^2(G)$, and it is easy to see that H extends to a bounded operator on $A_q(G)$. The corresponding result for $q > 2$ is not known. Clearly H extends to a bounded operator on A_q if and only if H extends to a bounded operator from $A_q(G)$ to $L^1(G)$.

In this paper we use Rudin-Shapiro polynomials (see [1] or [2]) to study the conjugation operator from $A_q(G)$ to Orlicz spaces $L^\phi(G)$. We recall the definition of Orlicz spaces below.

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We say that ϕ is a Young's function if ϕ is an increasing, continuous, convex function on $\mathbb{R}^+$ such that

$$\lim_{t\to 0}\frac{\phi(t)}{t}=0 \quad\text{and}\quad \lim_{t\to\infty}\frac{\phi(t)}{t}=\infty.$$

The Orlicz space $L^\phi(G)$ is the class of complex-valued measurable functions on G such that for some $\lambda>0$

$$\int_G \phi(|\lambda f(x)|)\,dx<\infty.$$

Then

$$\|f\|_{L^\phi}=\inf_{\lambda>0}\frac{1}{\lambda}\{N_\phi(\lambda f)+1\}$$

defines a Banach space norm on $L^\phi(G)$ [3]. The conditions on ϕ imply that $L^\phi(G)\subseteq L^1(G)$.

Theorem. *Let G be a compact, connected abelian group and P any fixed order on Γ. If $q>2$ and ϕ is a Young's function, then the conjugation operator H does not extend to a bounded operator from $A_q(G)$ to $L^\phi(G)$.*

We use Rudin-Shapiro polynomials (see [1]) to prove this theorem. These are described below.

Let $u\in\mathscr{T}$ be such that $E_0=\operatorname{supp}(\hat u)\subseteq P$. Let $f_0=g_0=u$. For $n\ge 0$, let $\gamma_n\notin(E_n-E_n)$, $\gamma_n\ge 0$, and define

$$f_{n+1}=f_n+\gamma_n g_n,\qquad g_{n+1}=f_n-\gamma_n g_n.$$

Then $\operatorname{supp}(\hat f_{n+1})=\operatorname{supp}(\hat g_{n+1})=E_{n+1}=E_n\cup(\gamma_n+E_n)\subseteq P$. For $\gamma\in\Gamma$, we have

$$\hat f_{n+1}(\gamma)=\hat f_n(\gamma)+\hat g_n(\gamma-\gamma_n).$$

Since $(\gamma_n+E_n)\cap E_n=\varnothing$,

$$\|\hat f_{n+1}\|_{l_r}^r=\|\hat f_n\|_{l_r}^r+\|\hat g_n\|_{l_r}^r,$$

similarly,

$$\|\hat g_{n+1}\|_{l_r}^r=\|\hat f_n\|_{l_r}^r+\|\hat g_n\|_{l_r}^r.$$

Therefore, by induction, we get

$$\|\hat f_{n+1}\|_{l_r}^r=2^n(\|\hat f_0\|_{l_r}^r+\|\hat g_0\|_{l_r}^r)=2^{n+1}\|\hat u\|_{l_r}^r.$$

Hence

$$\|\hat f_n\|_{l_r}=2^{n/r}\|\hat u\|_{l_r}\quad\text{and}\quad\|\hat g_n\|_{l_r}=2^{n/r}\|\hat u\|_{l_r}. \tag{1}$$

Next, using elementary algebra and induction on n, we have

$$\begin{aligned}|f_{n+1}(x)|^2+|g_{n+1}(x)|^2&=2(|f_n(x)|^2+|g_n(x)|^2)\\&=2^{n+1}(|f_0(x)|^2+|g_0(x)|^2)=2^{n+2}|u(x)|^2.\end{aligned}$$

Therefore,

$$|f_n(x)|\le 2^{(n+1)/2}|u(x)|\quad\text{and}\quad|g_n(x)|\le 2^{(n+1)/2}|u(x)|. \tag{2}$$

Further, we also have

$$|f_n(x)|+|g_n(x)|\ge(|f_n(x)|^2+|g_n(x)|^2)^{1/2}=2^{(n+1)/2}|u(x)|. \tag{3}$$

For a Young's function ϕ, define

$$h_n = \begin{cases} f_n & \text{if } \|f_n\|_{L^\phi} \geq \|g_n\|_{L^\phi}, \\ g_n & \text{otherwise.} \end{cases}$$

It is now easy to see that

$$\|\hat{h}_n\|_{l_r} = 2^{n/r}\|\hat{u}\|_{l_r} \tag{4}$$

and

$$\|h_n\|_{L^\phi} \leq 2^{(n+1)/2}\|u\|_{L^\phi}. \tag{5}$$

Also,

$$\|h_n\|_{L^\phi} = \max(\|f_n\|_{L^\phi}, \|g_n\|_{L^\phi}) \geq \tfrac{1}{2}(\|f_n\|_{L^\phi} + \|g_n\|_{L^\phi}).$$

Hence, from (3)

$$\|h_n\|_{L^\phi} \geq 2^{(n-1)/2}\|u\|_{L^\phi}. \tag{6}$$

Note that inequalities (5) and (6) also hold for the L^1-norm.

Proof of the theorem. Consider the projection operator defined on trigonometric polynomials as

$$S(f) = \sum_{\gamma \geq 0} \hat{f}(\gamma)\gamma, \qquad f \in \mathscr{T}.$$

Then $Hf = -i(2S(f) - f - \hat{f}(0))\ \forall f \in \mathscr{T}$. Clearly H extends to a bounded operator from A_q to L^ϕ if and only if S extends to a bounded operator from A_q to L^ϕ. Since $\phi(t)/t \to \infty$ as $t \to \infty$, there exists a sequence $(v_n) \subseteq \mathscr{T}$ with $\operatorname{supp}(\hat{v}_n) \subseteq P$ such that

$$\|v_n\|_{L^1} \to 0 \quad \text{and} \quad \|v_n\|_{L^\phi} \to \infty \qquad \text{as } n \to \infty.$$

Now let m_n be an increasing sequence of integers such that $|\operatorname{supp}(\hat{v}_n)| \leq m_n$, and let $u_n = v_n/2^{m_n/2}$. For each u_n, construct a sequence of Rudin-Shapiro polynomials $\{h_m^n\}_{m=1}^\infty$ as above with $\operatorname{supp}\{h_m^n\}^\wedge \subseteq P$. Now define $h_n = h_{m_n}^n$. Then $Sh_n = h_n$. From (4) we have

$$\begin{aligned} \|\hat{h}_n\|_{l_q} &= 2^{m_n(1/q-1/2)}\|\hat{v}_n\|_{l_q} \leq 2^{m_n(1/q-1/2)} m_n^{1/q}\|\hat{v}_n\|_{l_\infty} \\ &\leq 2^{m_n(1/q-1/2)} m_n^{1/q}\|v_n\|_{L^1}. \end{aligned}$$

Also from (5) and (6) we have

$$\|h_n\|_{L^1} \leq 2^{1/2}\|v_n\|_{L^1} \quad \text{and} \quad \|h_n\|_{L^\phi} \geq (2)^{-1/2}\|v_n\|_{L^\phi}.$$

Hence $\|h_n\|_{A_q} \to 0$ and $\|h_n\|_{L^\phi} = \|Sh_n\|_{L^\phi} \to \infty$ as $n \to \infty$. This proves that S does not extend to a bounded operator from A_q to L^ϕ.

Corollary. *If $q > 2$ and ϕ is any Young's function, then $A_q \not\subseteq L^\phi$.*

Remark. In particular if $\phi(t) = t(\log^+ t)$, we see that there exists $f \in A_q$ such that $f \notin L(\log L)$. We do not know whether there exists such a function which, in addition, is nonnegative. The implication of this, for the circle group $\mathbb{T}$ would be that $Hf \notin L^1(\mathbb{T})$ (using Theorem 2.10, Chapter VII of [6]).

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