PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 1, May 1994

A GENERALIZATION OF THE SHIMIZU-LEUTBECHER AND JORGENSEN INEQUALITIES TO MÖBIUS TRANSFORMATIONS IN $\mathbf{R}^N$

SA'AR HERSONSKY

(Communicated by Clifford J. Earle)

ABSTRACT. We give a generalization of the Shimizu-Leutbecher inequality and a partial generalization of the Jorgensen inequality to Möbius transformations in $\mathbf{R}^N$ using the Clifford algebra and the Vahlen group.

1. INTRODUCTION

Everyone who learns about Kleinian groups is familiar with the following Shimizu-Leutbecher inequality [S]: Let f and g be elements of $\mathrm{SL}(2, \mathbf{C})$, where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $c \neq 0$, and $f = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Suppose that $\langle f, g \rangle$ is discrete. Then $|c| \geq 1$.

Jorgensen generalized this inequality and obtained the following inequality [J]: Let f, g be elements of $\mathrm{SL}(2, \mathbf{C})$, and suppose that $\langle f, g \rangle$ is a discrete nonelementary group. Then $|\operatorname{tr}^2 f - 4| + |\operatorname{tr}(fgf^{-1}g^{-1}) - 2| \geq 1$.

This reduces to the Shimizu-Leutbecher inequality when f is parabolic. These two inequalities, which can be interpreted geometrically in terms of Möbius transformations defined by matrices in $\mathrm{SL}(2, \mathbf{C})$, are of major importance in the general theory of Kleinian groups.

The proofs of the Shimizu-Leutbecher and Jorgensen inequalities are based on manipulations of 2×2 matrices. It was shown by Vahlen [Va] that Möbius transformations in $\mathbf{R}^N$ can be defined by 2×2 matrices whose entries are Clifford numbers. In this paper we exploit this idea to obtain a natural generalization of the Shimizu-Leutbecher inequality to Möbius transformations in $\mathbf{R}^N$ by directly imitating the proof for $\mathrm{SL}(2, \mathbf{C})$. In the same way we obtain a natural partial generalization of the Jorgensen inequality in the case where f and $fgf^{-1}g^{-1}$ are hyperbolic.

Our generalization of the Shimizu-Leutbecher inequality turns out to be equivalent to the generalization given by Wielenberg in Proposition 4 of [Wie]. (The connection between our version of the result and Wielenberg's version is elucidated in [Her].) It may be that our partial generalization of the Jorgensen inequality can be proved from a point of view similar to that of [Wie]. However, the approach based on Clifford matrices has the advantage that it provides a

Received by the editors September 17, 1991 and, in revised form, September 2, 1992.
1991 *Mathematics Subject Classification*. Primary 30F40.

way of discovering both the statements and proofs of the appropriate generalizations by starting from the classical case of $\mathrm{SL}(2, \mathbf{C})$. This approach has already led to another generalization of the Shimizu-Leutbecher inequality and another partial generalization of the Jorgensen inequality to the symplectic group. This work will be presented in [Fri-Her]; it depends on thinking of the elements of the symplectic group as 2×2 matrices whose entries are themselves matrices.

We learned about Vahlen's work from a beautiful paper of Ahlfors [Ah1]. We recently learned that results similar to the ones in this paper, and based on Clifford matrices were obtained independently by [Wat].

For a different partial generalization of the Jorgensen inequality to higher dimensions, see [Mar].

In §2 we give a brief review of the basic properties of the Clifford algebra and the Vahlen group. The interested reader can find detailed proofs in [Ah1] and [Ah2]. We also show how to give an analogue of the familiar trace in $\mathrm{SL}(2, \mathbf{C})$ for 2×2 matrices whose entries are Clifford numbers. Our version of the Jorgensen inequality, like Jorgensen's version (but unlike the versions given in [Wat] and [Mar]), is stated in terms of the trace. The difficulty involved in finding a complete generalization of the Jorgensen inequality from our viewpoint is that there is no simple generalization of the trace for arbitrary Clifford matrices. A sophisticated generalization of the trace is given in [Wad], but we have not yet succeeded in exploiting it in the context of the Jorgensen inequality.

In §3 we will give the main results and their proofs. As we have stressed above, the ideas of the proofs are essentially the same as the standard proofs in $\mathrm{SL}(2, \mathbf{C})$. However, the computations are not as simple as in $\mathrm{SL}(2, \mathbf{C})$. One reason for this, which the reader should bear in mind, is that the Clifford algebra is not commutative.

In [Her], using the generalized Shimizu-Leutbecher inequality, we show how to get a general method for calculating a lower bound for the covolume of any discrete group of $\mathrm{Iso}^+(H^n)$ having parabolic elements. We get an explicit number in the case where $n = 4$ and the group is torsion-free.

2. The Clifford algebra, Clifford matrices, and hyperbolic Möbius transformations in $\mathbf{R}^N$

In this section we will briefly review some material on Clifford numbers and Clifford matrices that is treated in detail in [Ah1] and [Ah2]. Recall that the Clifford algebra A_N is the associative algebra over the reals generated by the elements $e_1, \dots, e_{N-1}$ subject to the two relations $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i \neq j$. Each element $a \in A_N$ can be written uniquely in the form $a = \sum a_v E_v$ where v runs over all ordered, finite multi-indices $v = v_1 \cdots v_p$ satisfying $0 < v_1 < \cdots < v_p < N$ and $E_v = e_{v_1} \cdots e_{v_p}$. The norm $|a|$ is defined by $|a|^2 = \sum a_v^2$.

The linear subspace of A_N spanned by $1, e_1, \dots, e_{N-1}$ is denoted by V^N, and its elements are called *vectors*. The nonzero elements of V^N are invertible in A_N and generate a subgroup of the group of units of A_N; this group is called the Clifford group and is denoted by Γ_N. We have $|ab| = |a|\,|b|$ for all $a, b \in \Gamma_N$.

There is a unique linear involution $a \mapsto a^*$ of A_N which is the identity on V^N and satisfies $(ab)^* = b^* a^*$. We have $|a^*| = |a|$ for all $a \in \Gamma_N$.

We denote by $\mathrm{SL}_+(\Gamma_N)$ the set of all 2×2 matrices $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying the following conditions:

(i) $a, b, c, d \in \Gamma_N \cup \{0\}$.
(ii) $\delta(g) = ad^* - bc^* = 1$.
(iii) $ab^*, cd^*, c^*a, d^*b \in V^N$.

It was Vahlen [Va] who first showed that $\mathrm{SL}_+(\Gamma_N)$ is a group under matrix multiplication. A direct computation shows that

$$g^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}.$$

Following Ahlfor's notation we will call the elements of $\mathrm{SL}_+(\Gamma_N)$ Clifford matrices of dimension N. We will also use PSL_+ for the quotient group $\bmod \pm I$.

$\mathrm{PSL}_+(\Gamma_N)$ is generated by the matrices

$$\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $a \in \Gamma_N$, $b \in V^N$.

The group $\mathrm{PSL}_+(\Gamma_N)$ acts on the set $\overline{V^N} = V^N \cup \infty$ according to the formula $g(x) = (ax+b)(cx+d)^{-1}$, suitably interpreted when $x = \infty$ or $(cx+d)^{-1} = 0$. We can identify V^N with $\mathbf{R}^N$ so that the action of PSL_+ is isomorphic to the action of the Möbius group.

A Clifford matrix (or the corresponding Möbius transformation) is called *hyperbolic* if it is conjugate to a matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$ where $\alpha \in \mathbf{R}$ and $\alpha \neq \pm 1$. (Thus according to our convention "hyperbolic transformation" is more special than "loxodromic transformation".) The reader should also remember that a hyperbolic transformation has only two fixed points in $\overline{\mathbf{R}^N}$.

It was Ahlfors [Ah2] who showed that the Clifford matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $c \neq 0$ is hyperbolic if and only if $c(ac^{-1}+c^{-1}d)$ is real and $(c(ac^{-1}+c^{-1}d))^2 > 4$. When this is so it is also true that $a + d^*$ is real and of absolute value > 2.

3. Results

We will first generalize the Shimizu-Leutbecher inequality to Möbius transformations in $\mathbf{R}^N$. For an equivalent result with a different proof see Proposition 4 in [Wie].

Theorem A. *Let* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *be an element of* $\mathrm{SL}_+(\Gamma_N)$ *where* $c \neq 0$. *Let* $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, *and suppose that* $\langle M, U \rangle$ *is discrete. Then* $|c| \geq 1$.

Proof. We set $M_0 = M$, and for $n \geq 0$ we set

$$M_{n+1} = M_n U M_n^{-1}. \tag{1}$$

We remind the reader that our multiplication *is not* commutative, so one has to carry out the calculations very carefully. We set

$$M_{n+1} = \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix},$$

and by (1) we get

$$
\begin{aligned}
a_{n+1} &= a_n d_n^* - (a_n + b_n)c_n^* = 1 - a_n c_n^*, \\
b_{n+1} &= -a_n b_n^* + (a_n + b_n)a_n^* = a_n a_n^*, \\
c_{n+1} &= c_n d_n^* - (c_n + d_n)c_n^* = -c_n c_n^*, \\
d_{n+1} &= -c_n b_n^* + (c_n + d_n)a_n^* = 1 + c_n a_n^*.
\end{aligned} \tag{2}
$$

Recall that the norm $|\cdot|$ is multiplicative on Γ_N and that $|a| = |a^*|$ for every $a \in \Gamma_N$. Combining these facts with (2) we get

$$
\begin{aligned}
|a_{n+1}| &= |1 - a_n c_n^*|, \\
|b_{n+1}| &= |a_n^2|, \\
|c_{n+1}| &= |c_n|^2, \\
|d_{n+1}| &= |1 + c_n a_n^*|.
\end{aligned} \tag{3}
$$

By induction we get

$$|c_n| = |c|^{2^n} \quad \text{for } n \geq 0. \tag{4}$$

In particular, if $|c| \neq 1$ we have

$$M_m = M_n \Rightarrow m = n. \tag{5}$$

Now it is clear by (3) that we will get an obvious contradiction to the assumption that $\langle M, U\rangle$ is discrete if we show that $\{a_n\}$ is a bounded sequence for $0 < |c| < 1$.

Lemma 1. *If* $0 < |c| < 1$ *then* $\{a_n\}$ *is a bounded sequence.*

Proof. By (4) and the hypothesis of the lemma we get

$$|c_n| \to 0 \tag{6}$$

for $n \to \infty$. In particular, $|c_n| < |c|$ for all $n > 0$.

Let us now choose $\tau > 0$ such that

$$\frac{1}{1 - |c|} < \tau \quad \text{and} \quad |a_0| = |a| < \tau. \tag{7}$$

Suppose by induction that

$$|a_n| < \tau. \tag{8}$$

Then using (6) and (7) we get

$$|a_{n+1}| = |1 - a_n c_n^*| \leq 1 + |a_n c_n^*| = 1 + |a_n|\,|c_n| < 1 + \tau|c| < \tau. \tag{9}$$

This completes the proof of the lemma.

If $|c| < 1$ then by (2), (4), and the lemma we can find a sequence in $\langle M, U\rangle$ that converges to M. We can now use (5) and the discreteness of $\langle M, U\rangle$ to get the desired contradiction. Q.E.D.

We point out that not every parabolic transformation is conjugate to U. An arbitrary parabolic transformation is conjugate in $\mathrm{PSL}_+(\Gamma_N)$ to a Euclidean motion without fixed points in $\mathbf{R}^N$. For $N > 2$ such a Euclidean motion is not necessarily a translation. However, a discrete group with finite covolume that contains a parabolic element may be shown to contain a conjugate of U.

This is what makes possible the application of Theorem A in [Her] to covolume estimates for discrete groups containing parabolics.

Our next task is to give a partial generalization of the Jorgensen inequality. We begin with some definitions and a lemma.

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a Clifford matrix which represents a hyperbolic Möbius transformation in $\mathbf{R}^N$. We define $\operatorname{tr}(g)$ to be $a + d^*$. A direct computation shows that $\operatorname{tr}(g)$ is invariant under conjugation by each of the generators listed in §2. We also use the fact that $a + d^*$ is real here.

Definition. A subgroup G of $\mathrm{SL}(\Gamma_N)$ is said to be *elementary* if there exists a finite G-orbit in $\mathbf{R}^{N+1}$.

Lemma 2. *Let f be a hyperbolic Möbius transformation in $\mathbf{R}^N$, and let μ: $\mathrm{SL}_+(\Gamma_N) \to \mathrm{SL}_+(\Gamma_N)$ be defined by*

$$\mu(g) = gfg^{-1}.$$

Suppose that there exists n such that $\mu^n(g) = f$. Then $\langle f, g\rangle$ is elementary.

Proof. This is proved in the same way as Theorem 5.1.4 in [Bea]. The proof given there for $\mathrm{SL}(2, \mathbf{C})$ makes crucial use of the fact that a nontrivial Möbius transformation has at most two fixed points. But a hyperbolic transformation has exactly two fixed points in any dimension.

We now come to our partial generalization of the Jorgensen inequality.

Theorem B. *Let f and g be Möbius transformations in $\mathbf{R}^N$ such that f and $fgf^{-1}g^{-1}$ are hyperbolic, and suppose that $\langle f, g\rangle$ is a discrete nonelementary group. Then*

$$|\operatorname{tr}^2(f) - 4| + |\operatorname{tr}(fgf^{-1}g^{-1}) - 2| \geq 1. \tag{10}$$

Proof. We select two matrices $\in \mathrm{SL}_+(\Gamma_N)$ representing f and g respectively, we set $B_0 = B$, and for $n \geq 0$ we define

$$B_{n+1} = B_n A B_n^{-1}. \tag{11}$$

In the notation of Lemma 2, we have $B_n = \mu^n(B)$. It is sufficient to show that if (10), fails, then for some n we have

$$B_n = A. \tag{12}$$

Without loss of generality we can assume that

$$A = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}, \quad \tau > 0, \ \tau \neq 1, \qquad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{13}$$

Let us denote by k the left-hand side of (10) and suppose that

$$k < 1. \tag{14}$$

A straightforward though tedious computation using the special properties of Clifford matrices shows that (14) leads to $k = (1 + |bc^*|)|\tau - \tau^{-1}|^2$.

Hence

$$(1 + |bc^*|)|\tau - \tau^{-1}|^2 < 1. \tag{15}$$

Relation (11) gives

$$
\begin{aligned}
a_{n+1} &= \tau a_n d_n^* - \tau^{-1} b_n c_n^*, \\
b_{n+1} &= b_n a_n^* (\tau^{-1} - \tau), \\
c_{n+1} &= c_n d_n^* (\tau - \tau^{-1}), \\
d_{n+1} &= \tau^{-1} d_n a_n^* - \tau c_n b_n^*.
\end{aligned} \tag{16}
$$

One can now use property (iii) of Clifford matrices to get

$$b_{n+1} c_{n+1} = -(1 + b_n c_n^*) b_n c_n^* (\tau - \tau^{-1})^2. \tag{17}$$

By a very simple induction argument we get

$$|b_n c_n| \le k^n |bc^*|. \tag{18}$$

We remind the reader that $bc \neq 0$; otherwise $\langle f, g \rangle$ would be elementary. We get

$$b_n c_n^* \le k^n |bc^*|, \quad \text{and therefore} \quad b_n c_n^* \to 0. \tag{19}$$

We now use $\delta(g) = a_n d_n^* - b_n c_n^* = 1$ to obtain

$$a_n d_n^* \to 1. \tag{20}$$

By (16) we have

$$a_{n+1} \to \tau, \qquad d_{n+1} \to \tau^{-1}. \tag{21}$$

Now

$$|b_n^{-1} b_{n+1}| = |a_n^* (\tau^{-1} - \tau)| = |a_n (\tau^{-1} - \tau)| \to |\tau(\tau^{-1} - \tau)| \le \sqrt{k}\tau. \tag{22}$$

So for sufficiently large n we have

$$|b_{n+1}/\tau^{n+1}| \le \sqrt{k} |b_n/\tau^n|. \tag{23}$$

It follows that

$$|b_n/\tau_n| \to 0. \tag{24}$$

In a very similar way one gets that

$$|c_n \tau^n| \to 0. \tag{25}$$

It follows that

$$A^{-n} B_{2n} A^n = \begin{pmatrix} a_{2n} & \tau^{-2n} b_{2n} \\ \tau^{2n} c_{2n} & d_{2n} \end{pmatrix} \to \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}. \tag{26}$$

Since we assumed that $\langle A, B \rangle$ is discrete, we must have $B_{2m} = A$ for some m. Q.E.D.

Acknowledgments

The author of this paper is a graduate student at the University of Illinois at Chicago under the supervision of Professor Peter Shalen.

The work on this paper began while the author was a pregraduate student at the Hebrew University in Jerusalem. He expresses deep thanks to Professors Peter Shalen and Alex Lubotzky for their advice and support. The author also thanks Professors H. Farkas, H. Furstenberg (Hebrew University), U. Srebro and D. Aharonov (Technion-Israel), for their patient listening and advice and finally, the referee for pointing out [Mar] and [Wie] and for other useful remarks.

REFERENCES

[Ah1] L. Ahlfors, *Möbius transformations and Clifford numbers*, Differential Geometry and Complex Analysis—in memory of H. E. Rauch (I. Chavel and H. Farkas, eds.), Springer-Verlag, New York, 1985.

[Ah2] ——, *On the fixed points of Möbius transformations in* $\mathbf{R}^N$, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985).

[Bea] A. F. Beardon, *The geometry of discrete groups*, Springer-Verlag, New York, 1983.

[Fri-Her] S. Friedland and S. Hersonsky, *Jorgensen inequality for discrete groups in normed algebras*, Duke Math. J. **69** (1993), 593–614.

[Her] S. Hersonsky, *Volume estimates for discrete groups of* $\mathrm{Iso}^+(H^n)$ *having parabolic elements*, Michigan Math. J. (to appear).

[J] T. Jorgensen, *On discrete groups of Möbius transformations*, Amer. J. Math. **98** (1976), 736–739.

[Mar] G. Martin, *On discrete Möbius groups in all dimensions*, Acta. Math. **163** (1989), 253–289.

[S] H. Shimizu, *On discontinuous groups operating on the product of half spaces*, Ann. of Math. (2) **77** (1963), 33–71.

[Leu] A. Leutbecher, *Uber spitzen diskontinuierlicher von lineargebrchener transformations*, Math. Zeit. **100** (1967), 183–200.

[Va] K. Vahlen, *Uber Bewegungen und complex Zahlen*, Math. Ann. **55** (1902), 585–593.

[Wad] M. Wada, *Conjugacy invariants of Möbius transformations*, Complex Variables Theory Appl. **15** (1990), 125–133.

[Wat] P. Waterman, *Möbius transformations in several dimensions*, preprint.

[Wie] N. J. Wielenberg, *Discrete Möbius groups*, Amer. J. Math. **99** (1977), 861–867.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO, ILLINOIS 60680

Current address: Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, Illinois 60680

E-mail address: `u21208@uicvm.uic.edu`