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ON THE MEDIANS OF GAMMA DISTRIBUTIONS AND AN EQUATION OF RAMANUJAN

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ABSTRACT. For $n \geq 0$, let $\lambda(n)$ denote the median of the $\Gamma(n+1, 1)$ distribution. We prove that $n + \frac{2}{3} < \lambda(n) \leq \min(n + \log 2, n + \frac{2}{3} + (2n+2)^{-1})$. These bounds are sharp. There is an intimate relationship between $\lambda(n)$ and an equation of Ramanujan. Based on this relationship, we derive the asymptotic expansion of $\lambda(n)$ as follows:

$$\lambda(n) = n + \frac{2}{3} + \frac{8}{405n} - \frac{64}{5103n^2} + \frac{2^7 \cdot 23}{3^9 \cdot 5^2 n^3} + \cdots .$$

Let median(Z_μ) denote the median of a Poisson random variable with mean μ, where the median is defined to be the least integer m such that $P(Z_\mu \leq m) \geq \frac{1}{2}$. We show that the bounds on $\lambda(n)$ imply

$$\mu - \log 2 \leq \text{median}(Z_\mu) < \mu + \tfrac{1}{3}.$$

This proves a conjecture of Chen and Rubin. These inequalities are sharp.

1. Introduction and statement of main results

For a continuous random variable X, the median is defined to be the least $x \in (-\infty, \infty)$ such that $P(X \leq x) = \frac{1}{2}$; and if X is an integer-valued random variable, the median of X is defined to be the least integer m such that $P(X \leq m) \geq \frac{1}{2}$. In either case, we denote the median of X by median(X). For $n \geq 0$, let $\lambda(n)$ denote the median of the $\Gamma(n+1, 1)$ distribution, the Gamma distribution with parameters $n+1$ and 1. In other words,

$$(1) \qquad \frac{1}{n!}\int_0^{\lambda(n)} t^n e^{-t} dt = \frac{1}{n!}\int_{\lambda(n)}^{\infty} t^n e^{-t} dt = \frac{1}{2}.$$

One of the main results in this paper is

Theorem 1. *For* $n \geq 0$,

$$(2) \qquad n + \tfrac{2}{3} < \lambda(n) \leq \min(n + \log 2, n + \tfrac{2}{3} + (2n+2)^{-1}).$$

These bounds are sharp.

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In this paper, we also consider the median of the Poisson distribution. Let Z_μ denote a Poisson random variable with mean μ. Then it will be shown in the next section that Theorem 1 implies the following

Theorem 2. *Let* $\mu \in (0, \infty)$. *Then*

$$\mu - \log 2 \leq \text{median}(Z_\mu) < \mu + \tfrac{1}{3}.$$

These bounds are best possible.

This proves Conjecture 1a of Chen and Rubin [2]. Based on a numerical study, Chen and Rubin were led to a stronger conjecture, which states that $\lambda(n) - n$ is decreasing in n. This still remains open. At the end of this section, we see that the coefficients of the first four terms of the asymptotic expansion of $\lambda(n-1) - (n-1)$ are positive. This strongly suggests that their conjecture is very likely to hold.

The work of this paper is also motivated by its connection with a well-known equation of Ramanujan. Ramanujan [7] and also his first letter to Hardy on January 16, 1913 contained the following statement: For $n \geq 1$, let $\theta(n)$ be defined as

$$\frac{e^n}{2} = \sum_{k=0}^{n-1} \frac{n^k}{k!} + \theta(n)\frac{n^n}{n!}; \tag{3}$$

then $\frac{1}{3} \leq \theta(n) \leq \frac{1}{2}$. He also gave the first four terms of the asymptotic expansion of $\theta(n)$:

$$\theta(n) = \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \frac{16}{8505n^3} + O\left(\frac{1}{n^4}\right). \tag{4}$$

This problem has attracted a lot of attention. For example, Szegö [9] and Watson [10] proved that $\theta(n)$ lies between $\frac{1}{3}$ and $\frac{1}{2}$, whereas further coefficients of the asymptotic expansion have been found by Bowman, Shenton, and Szekeres [1] and Marsaglia [6]. Knuth considered the asymptotic expansion of $P(Z_n \leq n-1)/P(Z_n = n)$ from which he gave another derivation of the asymptotic expansion of $\theta(n)$, though with fewer terms (see Knuth [5, pp. 112–118]). We find that there is an intimate relationship between $\lambda(n)$ and $\theta(n)$ (see the first paragraph of §3); namely,

$$1 - \theta(n) = \int_n^{\lambda(n)} t^{n-1}e^{-t}/n^{n-1}e^{-n}dt. \tag{5}$$

This will enable us to deduce bounds on $\theta(n)$ based on Theorem 1 and, conversely, asymptotic expansion of $\lambda(n)$ based on that of $\theta(n)$.

Theorem 3. *With* $\theta(n)$ *defined as above and* $n \geq 1$, *we have*

$$\frac{1}{3} - \frac{73}{162n} + \frac{19}{243n^2} < \theta(n) < \frac{1}{3} + \frac{4}{81n} - \frac{4}{243n^3}.$$

The upper bound in Theorem 3 is a decreasing function in n, and hence $\theta(n) < \frac{1}{3} + \frac{8}{243} = 0.36625\ldots$, which is an improvement of the upper bound $\frac{1}{2}$.

Theorem 4. *With $\lambda(n)$ defined as above, we have*

$$\lambda(n) = n + \frac{2}{3} + \frac{8}{405n} - \frac{64}{5103n^2} + \frac{2^7 \cdot 23}{3^9 \cdot 5^2 n^3} + O\left(\frac{1}{n^4}\right). \tag{6}$$

The first three terms of the asymptotic expansion can also be derived from Dinges [3, Proposition 3.6]. Our method here is different, and with more work many more terms can be derived. Doodson [4] considered

$$\frac{\operatorname{median}(X_n) - \operatorname{mode}(X_n)}{\operatorname{mean}(X_n) - \operatorname{mode}(X_n)} = \operatorname{median}(X_n) - (n-1),$$

where X_n denotes a Gamma random variable with parameters n and 1. Using our notation and results, this ratio is

$$\lambda(n-1) - (n-1) = \frac{2}{3} + \frac{8}{405n} + \frac{144}{3^6 \cdot 5 \cdot 7n^2} + \frac{2^3 \cdot 281}{3^9 \cdot 5^2 \cdot 7n^3} + \cdots.$$

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let X_{n+1} be $\Gamma(n+1, 1)$ distributed. For $x \in [\frac{2}{3}, 1]$, $n \geq 0$, let $a_n(x) = P(X_{n+1} \geq n + x)$ and $A_n(x) = (n+1)![a_{n+1}(x) - a_n(x)]$. For $n \geq 0$ and $t > 0$, let $\psi_n(t) = t^{n+1}e^{-t}$. Therefore,

$$\begin{aligned} A_n(x) &= \int_{n+x+1}^{\infty} t^{n+1}e^{-t}\,dt - (n+1)\int_{n+x}^{\infty} t^n e^{-t}\,dt \\ &= (n+x)^{n+1}e^{-(n+x)} - \int_{n+x}^{n+x+1} t^{n+1}e^{-t}\,dt \\ &= \int_{n+x}^{n+x+1} [\psi_n(n+x) - \psi_n(t)]\,dt. \end{aligned}$$

We shall show at the end of the proof that

$$A_n(\log 2) > 0, \qquad n \geq 2; \tag{7}$$

$$A_n(\tfrac{2}{3} + (2n+2)^{-1}) > 0, \qquad n \geq 1; \tag{8}$$

$$A_n(\tfrac{2}{3}) < 0, \qquad n \geq 1. \tag{9}$$

By the Central Limit Theorem, we have $a_n(x) \longrightarrow \frac{1}{2}$ as $n \longrightarrow \infty$. From (9) we have $a_n(\frac{2}{3})$ is a strictly decreasing sequence; therefore, $a_n(\frac{2}{3}) > \frac{1}{2}$. Hence, $\lambda(n) > n + \frac{2}{3}$, $n \geq 1$. Direct computation shows that $\lambda(0) = \log 2 > \frac{2}{3}$, establishing inequality (2) in the case $n = 0$. The second inequality is proved in the same way. To prove (7)–(9), we write $A_n(x) = B_n(x) + E_n(x)$, where $B_n(x)$ is due to the Taylor expansion of $\psi_n(x)$ at $n+1$ up to the third derivative and with $E_n(x)$ the error term. It is not difficult to derive that

$$B_n(x) = \left\{ \frac{(3x-2)}{6(n+1)} - \frac{6x^2-8x+3}{12(n+1)^2} \right\} \psi_n(n+1),$$

and there exist $\xi \in [n+x, n+1]$ and $\xi(t)$ lying between $n+1$ and t such that

$$E_n(x) = \frac{(1-x)^4}{4!}\psi_n^{(4)}(\xi) - \int_{n+x}^{n+x+1} \frac{(t-n-1)^4}{4!}\psi_n^{(4)}(\xi(t))\,dt.$$

Write $\psi_n^{(k)}(t) = w_{k,n}(t)\psi_n(t)/t^k$ and $u_{k,n}(s) = w_{k,n}(n+1+s)$ for $s \in [-(1-x), x] \subseteq [-1, 1]$. It is straightforward to verify that $u_{4,n}(s) = s^4 + (n+1)[3(n-1) - 8s - 6s^2]$, it is decreasing in s, and $u_{4,n}(s) \geq u_{4,n}(1) > 0$ when $n \geq 6$. For $n \geq 6$, this implies that $\psi_n^{(4)}(\xi(t)) > 0$; hence, $E_n(x)$ is bounded above by

$$\frac{(1-x)^4 u_{4,n}(-1)}{24(n+x)^4}\psi_n(n+1).$$

Therefore, $A_n(\frac{2}{3}) < B_n(\frac{2}{3}) + \psi_n(n+1)/[648(n+1)^2] < 0$, proving (9) for $n \geq 6$. Remaining cases can be verified by direct calculation.

Next, for $n \geq 6$, $E_n(x)$ is bounded below by

$$\frac{-[x^5 + (1-x)^5]u_{4,n}(-1)}{120(n+x)^4}\psi_n(n+1).$$

Since the leading term of $A_n(\log 2)$ is $(3\log 2 - 2)/[6(n+1)]$, it is apparent that $A_n(\log 2) > 0$ for n sufficiently large. Indeed, it can be verified to be positive for $n \geq 6$. Also, we have $B_n(\frac{2}{3} + \frac{1}{2(n+1)}) = (\frac{2}{9(n+1)^2} - \frac{1}{8(n+1)^4})\psi_n(n+1)$ and $E_n(\frac{2}{3} + \frac{1}{2(n+1)}) > \frac{-3}{40(n+1)^2}\psi_n(n+1)$. Hence, $A_n(\frac{2}{3} + \frac{1}{2(n+1)}) > 0$. Remaining cases can be verified by direct calculation. Since $\lambda(0) = \log 2$ and $\lambda(n) - n \to \frac{2}{3}$ as $n \to \infty$, the bounds are sharp. This completes the proof of Theorem 1. □

Remark 1. The lower bound in (2) and hence the upper bound in Theorem 2 were proved by an ingenious and yet elementary method in Chen and Rubin [2].

Proof of Theorem 2. Making use of the Poisson-Gamma relation, which states that for $n \geq 0$, $\mu > 0$

$$P(Z_\mu \leq n) = P(X_{n+1} \geq \mu), \tag{10}$$

we see that $P(Z_{\lambda(n)} \leq n) = \frac{1}{2}$ and, for $\lambda(n) < \mu \leq \lambda(n+1)$, median$(Z_\mu)$ $= n+1$. Since $\lambda(n+1) - \lambda(n) > n+1+\frac{2}{3} - (n+\log 2) > 0.97$, for any $\mu \in (\log 2, \infty)$, there exists $n \geq 0$ such that $\lambda(n) < \mu \leq \lambda(n+1)$. Therefore,

$$\text{median}(Z_\mu) - \mu = n+1-\mu \geq n+1-\lambda(n+1) \geq -\log 2$$

and

$$\text{median}(Z_\mu) - \mu = n+1-\mu < n+1-\lambda(n) < \tfrac{1}{3}.$$

For $\mu \in (0, \log 2]$, median$(Z_\mu) = 0$, so $-\log 2 \leq$ median(Z_μ) with equality holds when $\mu = \log 2$. Let $\mu = \lambda(n) + \epsilon$, where ϵ is an arbitrarily small positive number. Then median$(Z_\mu) - \mu = n+1-\lambda(n) - \epsilon = \frac{1}{3} - \epsilon + O(n^{-1})$.

This shows that the second inequality is sharp, and this completes the proof of Theorem 2. □

3. Proofs of Theorems 3 and 4

It is obvious that equation (4) is equivalent to

$$P(Z_n \leq n) - [1 - \theta(n)]P(Z_n = n) = \tfrac{1}{2}.$$

Recalling the Poisson-Gamma relation (see equation (10)) and the definitions of ψ_n and $\lambda(n)$, we obtain (5).

Proof of Theorem 3. Recall the definition of $\psi_n(t)$. Then

$$1 - \theta(n) < \int_n^{n+2/3+(2n)^{-1}} \psi_{n-1}(t)/\psi_{n-1}(n)\, dt.$$

Expanding $\psi_{n-1}(t)$ at $t = n$ up to the second derivative and estimating the error term as in the proof of Theorem 1, we can show that $\psi_{n-1}(t)/\psi_{n-1}(n)$ is bounded above by

$$\begin{aligned} 1 - \frac{(t-n)^2}{2n} &+ \frac{(t-n)^3}{6n^3} \max_{0 \leq s \leq 2/3+(2n)^{-1}} ((2+3s)n - s^3) \\ &\leq 1 - \frac{(t-n)^2}{2n} + \frac{(t-n)^3}{6n^3}\left(4n + \frac{3}{2}\right), \end{aligned}$$

from which we obtain the lower bound for $\theta(n)$. As shown in the proof of Theorem 1, for $n \geq 6$, $\psi_{n-1}^{(4)}(t) > 0$ on $[n, n + \frac{2}{3}]$. Therefore, $\psi_{n-1}'''(\xi) \geq \psi_{n-1}'''(n)$. It now follows that

$$\frac{\psi_{n-1}(t)}{\psi_{n-1}(n)} \geq 1 - \frac{(t-n)^2}{2n} + \frac{(t-n)^3}{3n^2}.$$

For $n \geq 6$, this inequality and Theorem 1 imply

$$1 - \theta(n) \geq \int_n^{n+2/3} \frac{\psi_{n-1}(t)}{\psi_{n-1}(n)}\, dt \geq \frac{2}{3} - \frac{4}{81n} + \frac{4}{243n^2},$$

and the upper bound for $\theta(n)$ follows. It can be verified that the upper bound still holds for $1 \leq n \leq 5$. This completes the proof of Theorem 3. □

Proof of Theorem 4. Using the Taylor expansion of ψ_{n-1} at $t = n$, we have

$$1 - \theta(n) = \sum_{k=0}^{N-1} \frac{\psi_{n-1}^{(k)}(n)}{\psi_{n-1}(n)} \frac{(\lambda(n) - n)^{k+1}}{(k+1)!} + R_N.$$

We estimate the error term R_N as follows. Recall that

$$\psi_{n-1}^{(N)}(t) = w_{N,n-1}(t)\psi_{n-1}(t)/t^N.$$

It is not difficult to show by induction that $\| w_{N,n-1} \| = \max\{|w_{N,n-1}(t)| : n \leq t \leq n + \log 2\} \leq N! n^{N/2}$. Therefore,

$$|R_N| = \left| \int_n^{\lambda(n)} \frac{(t-n)^N}{N!} \frac{\psi_{n-1}^{(N)}(\xi(t))}{\psi_{n-1}(n)} \, dt \right| \leq \frac{(\lambda(n)-n)^{N+1}}{(N+1)!} \frac{\| w_{N,n-1} \|}{n^N}$$
$$\leq \frac{(\lambda(n)-n)^{N+1}}{(N+1) n^{N/2}} \to 0 \quad \text{as } N \to \infty.$$

Let $a_k = \psi_{n-1}^{(k)}(n)/\psi_{n-1}(n)$ for $k \geq 0$. It is easy to verify that $a_0 = 1$, $a_1 = 0$, and, for $k \geq 2$, $a_k = -(k-1)[a_{k-1} + a_{k-2}]/n$. Then we obtain

$$1 - \theta(n) = \sum_{k=0}^{\infty} a_k \frac{(\lambda(n)-n)^{k+1}}{(k+1)!}.$$

From this we can derive the asymptotic expansion of $\lambda(n)$. This finishes the proof. □

4. Remarks

It is not difficult to see that λ median$(\Gamma(\alpha, \lambda))$ = median$(\Gamma(\alpha, 1))$. Therefore, we have the following corollary from Theorem 1.

Corollary 5. *For $n \geq 1$ and $\lambda > 0$, we have*

$$\frac{-1}{3\lambda} \leq \text{median}(\Gamma(n, \lambda)) - \text{mean}(\Gamma(n, \lambda)) \leq \frac{-1 + \log 2}{\lambda}.$$

The bounds are best possible.

Using the fact that $\chi_n^2 = \Gamma(\frac{n}{2}, \frac{1}{2})$, we have

Corollary 6. *For $m \geq 1$,*

$$-\tfrac{2}{3} \leq \text{median}(\chi_{2m}^2) - \text{mean}(\chi_{2m}^2) \leq 2(-1 + \log 2).$$

The bounds are best possible.

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