ON THE MEDIANS OF GAMMA DISTRIBUTIONS
AND AN EQUATION OF RAMANUJAN

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Abstract. For \( n \geq 0 \), let \( \lambda(n) \) denote the median of the \( \Gamma(n + 1, 1) \) distribution. We prove that
\[ n + \frac{2}{3} < \lambda(n) \leq \min(n + \log 2, n + \frac{2}{3} + (2n + 2)^{-1}). \]
These bounds are sharp. There is an intimate relationship between \( \lambda(n) \) and an equation of Ramanujan. Based on this relationship, we derive the asymptotic expansion of \( \lambda(n) \) as follows:
\[ \lambda(n) = n + \frac{2}{3} + \frac{8}{405n} - \frac{64}{5103n^2} + \frac{27 \cdot 23}{39 \cdot 5^2 n^3} + \cdots. \]

Let \( \text{median}(Z_\mu) \) denote the median of a Poisson random variable with mean \( \mu \), where the median is defined to be the least integer \( m \) such that \( P(Z_\mu \leq m) \geq \frac{1}{2} \). We show that the bounds on \( \lambda(n) \) imply
\[ \mu - \log 2 \leq \text{median}(Z_\mu) < \mu + \frac{1}{3}. \]
This proves a conjecture of Chen and Rubin. These inequalities are sharp.

1. Introduction and statement of main results

For a continuous random variable \( X \), the median is defined to be the least \( x \in (-\infty, \infty) \) such that \( P(X \leq x) = \frac{1}{2} \); and if \( X \) is an integer-valued random variable, the median of \( X \) is defined to be the least integer \( m \) such that \( P(X \leq m) \geq \frac{1}{2} \). In either case, we denote the median of \( X \) by \( \text{median}(X) \). For \( n \geq 0 \), let \( \lambda(n) \) denote the median of the \( \Gamma(n + 1, 1) \) distribution, the Gamma distribution with parameters \( n + 1 \) and \( 1 \). In other words,
\[ \frac{1}{n!} \int_0^{\lambda(n)} t^n e^{-t} dt = \frac{1}{n!} \int_{\lambda(n)}^{\infty} t^n e^{-t} dt = \frac{1}{2}. \]

One of the main results in this paper is

Theorem 1. For \( n \geq 0 \),
\[ n + \frac{2}{3} < \lambda(n) \leq \min(n + \log 2, n + \frac{2}{3} + (2n + 2)^{-1}). \]
These bounds are sharp.
In this paper, we also consider the median of the Poisson distribution. Let \( Z_\mu \) denote a Poisson random variable with mean \( \mu \). Then it will be shown in the next section that Theorem 1 implies the following

**Theorem 2.** Let \( \mu \in (0, \infty) \). Then
\[
\mu - \log 2 \leq \text{median}(Z_\mu) < \mu + \frac{1}{2}.
\]

These bounds are best possible.

This proves Conjecture 1a of Chen and Rubin [2]. Based on a numerical study, Chen and Rubin were led to a stronger conjecture, which states that \( \lambda(n) - n \) is decreasing in \( n \). This still remains open. At the end of this section, we see that the coefficients of the first four terms of the asymptotic expansion of \( \lambda(n - 1) - (n - 1) \) are positive. This strongly suggests that their conjecture is very likely to hold.

The work of this paper is also motivated by its connection with a well-known equation of Ramanujan. Ramanujan [7] and also his first letter to Hardy on January 16, 1913 contained the following statement: For \( n \geq 1 \), let \( \theta(n) \) be defined as

\[
\frac{e^n}{2} = \sum_{k=0}^{n-1} \frac{n^k}{k!} + \theta(n) \frac{n^n}{n!};
\]

then \( \frac{1}{3} \leq \theta(n) \leq \frac{1}{2} \). He also gave the first four terms of the asymptotic expansion of \( \theta(n) \):

\[
\theta(n) = \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \frac{16}{8505n^3} + O\left(\frac{1}{n^4}\right).
\]

This problem has attracted a lot of attention. For example, Szegö [9] and Watson [10] proved that \( \theta(n) \) lies between \( \frac{1}{3} \) and \( \frac{1}{2} \), whereas further coefficients of the asymptotic expansion have been found by Bowman, Shenton, and Szekeres [1] and Marsaglia [6]. Knuth considered the asymptotic expansion of \( P(Z_n \leq n - 1)/P(Z_n = n) \) from which he gave another derivation of the asymptotic expansion of \( \theta(n) \), though with fewer terms (see Knuth [5, pp. 112–118]). We find that there is an intimate relationship between \( \lambda(n) \) and \( \theta(n) \) (see the first paragraph of §3); namely,

\[
1 - \theta(n) = \int_n^{\lambda(n)} t^{-1}e^{-t}/n^{n-1}e^{-ndt}.
\]

This will enable us to deduce bounds on \( \theta(n) \) based on Theorem 1 and, conversely, asymptotic expansion of \( \lambda(n) \) based on that of \( \theta(n) \).

**Theorem 3.** With \( \theta(n) \) defined as above and \( n \geq 1 \), we have
\[
\frac{1}{3} - \frac{73}{162n} + \frac{19}{243n^2} < \theta(n) < \frac{1}{3} + \frac{4}{81n} - \frac{4}{243n^3}.
\]

The upper bound in Theorem 3 is a decreasing function in \( n \), and hence \( \theta(n) < \frac{1}{3} + \frac{8}{243} = 0.36625 \ldots \), which is an improvement of the upper bound \( \frac{1}{2} \).
Theorem 4. With \( \lambda(n) \) defined as above, we have

\[
\lambda(n) = n + \frac{2}{3} + \frac{8}{405n} - \frac{64}{5103n^2} + \frac{2^7 \cdot 23}{3^9 \cdot 5^2 n^3} + O\left(\frac{1}{n^4}\right).
\]

The first three terms of the asymptotic expansion can also be derived from Dinges [3, Proposition 3.6]. Our method here is different, and with more work many more terms can be derived. Doodson [4] considered

\[
\frac{\text{median}(X_n) - \text{mode}(X_n)}{\text{mean}(X_n) - \text{mode}(X_n)} = \frac{1}{\lambda(n - 1)} - (n - 1),
\]

where \( X_n \) denotes a Gamma random variable with parameters \( n \) and 1. Using our notation and results, this ratio is

\[
\lambda(n - 1) - (n - 1) = \frac{2}{3} + \frac{8}{405n} + \frac{144}{3^6 \cdot 5 \cdot 7n^2} + \frac{23 \cdot 281}{3^9 \cdot 5^2 \cdot 7n^3} + \cdots.
\]

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let \( X_{n+1} \) be \( \Gamma(n+1, 1) \) distributed. For \( x \in [\frac{1}{3}, 1], \ n \geq 0 \), let \( a_n(x) = P(X_{n+1} \geq n + x) \) and \( A_n(x) = (n + 1)! [a_{n+1}(x) - a_n(x)] \). For \( n \geq 0 \) and \( t > 0 \), let \( \psi_n(t) = t^{n+1} e^{-t} \). Therefore,

\[
A_n(x) = \int_{n+x+1}^{\infty} t^{n+1} e^{-t} \, dt - (n + 1) \int_{n+x}^{\infty} t^n e^{-t} \, dt
\]

\[
= (n + x)^{n+1} e^{-(n+x)} - \int_{n+x}^{n+x+1} t^{n+1} e^{-t} \, dt
\]

\[
= \int_{n+x}^{n+x+1} [\psi_n(n + x) - \psi_n(t)] \, dt.
\]

We shall show at the end of the proof that

\[
A_n(\log 2) > 0, \quad n \geq 2;
\]

\[
A_n(\frac{2}{3} + (2n + 2)^{-1}) > 0, \quad n \geq 1;
\]

\[
A_n(\frac{1}{3}) < 0, \quad n \geq 1.
\]

By the Central Limit Theorem, we have \( a_n(x) \longrightarrow \frac{1}{2} \) as \( n \longrightarrow \infty \). From (9) we have \( a_n(\frac{2}{3}) \) is a strictly decreasing sequence; therefore, \( a_n(\frac{2}{3}) > \frac{1}{2} \). Hence, \( \lambda(n) > n + \frac{2}{3}, \quad n \geq 1 \). Direct computation shows that \( \lambda(0) = \log 2 > \frac{2}{3} \), establishing inequality (2) in the case \( n = 0 \). The second inequality is proved in the same way. To prove (7)–(9), we write \( A_n(x) = B_n(x) + E_n(x) \), where \( B_n(x) \) is due to the Taylor expansion of \( \psi_n(x) \) at \( n + 1 \) up to the third derivative and
with $E_n(x)$ the error term. It is not difficult to derive that

$$B_n(x) = \left\{ \frac{(3x - 2)}{6(n + 1)} - \frac{6x^2 - 8x + 3}{12(n + 1)^2} \right\} \psi_n(n + 1),$$

and there exist $\xi \in [n + x, n + 1]$ and $\xi(t)$ lying between $n + 1$ and $t$ such that

$$E_n(x) = (1 - x)^4 \psi_n^{(4)}(\xi) - \int_{n + x}^{t} \frac{(t - n - 1)^4}{4!} \psi_n^{(4)}(\xi(t)) \, dt.$$

Write $\psi_n^{(k)}(t) = w_{k,n}(t)\psi_n(t)/t^k$ and $u_{k,n}(s) = w_{k,n}(n + 1 + s)$ for $s \in \left[-(1 - x), x\right] \subseteq [-1, 1]$. It is straightforward to verify that $u_{4,n}(s) = s^4 + (n + 1)[3(n - 1) - 8s - 6s^2]$, it is decreasing in $s$, and $u_{4,n}(s) \geq u_{4,n}(1) > 0$ when $n \geq 6$. For $n \geq 6$, this implies that $\psi_n^{(4)}(\xi(t)) > 0$; hence, $E_n(x)$ is bounded above by

$$\frac{(1-x)^4 u_{4,n}(1)}{24(n+x)^4} \psi_n(n+1).$$

Therefore, $A_n(\frac{2}{3}) < B_n(\frac{2}{3}) + \psi_n(n+1)/[648(n+1)^2] < 0$, proving (9) for $n \geq 6$. Remaining cases can be verified by direct calculation.

Next, for $n \geq 6$, $E_n(x)$ is bounded below by

$$\frac{-[x^5 + (1-x)^5]u_{4,n}(1)}{120(n+x)^4} \psi_n(n+1).$$

Since the leading term of $A_n(\log 2)$ is $(3\log 2 - 2)/[6(n+1)]$, it is apparent that $A_n(\log 2) > 0$ for $n$ sufficiently large. Indeed, it can be verified to be positive for $n \geq 6$. Also, we have $B_n(\frac{2}{3} + \frac{1}{2(n+1)}) = (\frac{2}{9(n+1)^2} - \frac{1}{8(n+1)^2}) \psi_n(n+1)$ and $E_n(\frac{2}{3} + \frac{1}{2(n+1)}) > \frac{3}{40(n+1)^2} \psi_n(n+1)$. Hence, $A_n(\frac{2}{3} + \frac{1}{2(n+1)}) > 0$. Remaining cases can be verified by direct calculation. Since $\lambda(0) = \log 2$ and $\lambda(n) - n \rightarrow \frac{2}{3}$ as $n \rightarrow \infty$, the bounds are sharp. This completes the proof of Theorem 1.

**Remark 1.** The lower bound in (2) and hence the upper bound in Theorem 2 were proved by an ingenious and yet elementary method in Chen and Rubin [2].

**Proof of Theorem 2.** Making use of the Poisson-Gamma relation, which states that for $n \geq 0$, $\mu > 0$

$$P(Z_\mu \leq n) = P(X_{n+1} \geq \mu),$$

we see that $P(Z_{\lambda(n)} \leq n) = \frac{1}{2}$ and, for $\lambda(n) < \mu \leq \lambda(n + 1)$, median($Z_\mu$) = $n + 1$. Since $\lambda(n + 1) - \lambda(n) > n + 1 + \frac{2}{3} - (n + \log 2) > 0.97$, for any $\mu \in (\log 2, \infty)$, there exists $n \geq 0$ such that $\lambda(n) < \mu \leq \lambda(n + 1)$. Therefore, $\lambda(n) - \mu \leq n + 1 - \lambda(n + 1) > -\log 2$

and

$$\text{median}(Z_\mu) - \mu = n + 1 - \mu < n + 1 - \lambda(n) \leq \frac{2}{3}.$$

For $\mu \in (0, \log 2)$, $\text{median}(Z_\mu) = 0$, so $-\log 2 \leq \text{median}(Z_\mu)$ with equality holds when $\mu = \log 2$. Let $\mu = \lambda(n) + \epsilon$, where $\epsilon$ is an arbitrarily small positive number. Then $\text{median}(Z_\mu) - \mu = n + 1 - \lambda(n) - \epsilon = \frac{1}{2} - \epsilon + O(n^{-1})$. 

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This shows that the second inequality is sharp, and this completes the proof of Theorem 2. □

3. PROOFS OF THEOREMS 3 AND 4

It is obvious that equation (4) is equivalent to

\[ P(Z_n \leq n) - [1 - \theta(n)]P(Z_n = n) = \frac{1}{2}. \]

Recalling the Poisson-Gamma relation (see equation (10)) and the definitions of \( \psi_n \) and \( \lambda(n) \), we obtain (5).

**Proof of Theorem 3.** Recall the definition of \( \psi_n(t) \). Then

\[ 1 - \theta(n) < \int_n^{n+2/3+(2n)^{-1}} \frac{\psi_{n-1}(t)}{\psi_{n-1}(n)} \, dt. \]

Expanding \( \psi_{n-1}(t) \) at \( t = n \) up to the second derivative and estimating the error term as in the proof of Theorem 1, we can show that \( \psi_{n-1}(t)/\psi_{n-1}(n) \) is bounded above by

\[
1 - \frac{(t-n)^2}{2n} + \frac{(t-n)^3}{6n^3} \max_{0 \leq s \leq 2/3+(2n)^{-1}} ((2 + 3s)n - s^3) \\
\leq 1 - \frac{(t-n)^2}{2n} + \frac{(t-n)^3}{6n^3} \left( 4n + \frac{3}{2} \right),
\]

from which we obtain the lower bound for \( \theta(n) \). As shown in the proof of Theorem 1, for \( n \geq 6 \), \( \psi_{n-1}^{(4)}(t) > 0 \) on \( [n, n + \frac{2}{3}] \). Therefore, \( \psi_{n-1}''''(\xi) \geq \psi_{n-1}''''(n) \). It now follows that

\[
\frac{\psi_{n-1}(t)}{\psi_{n-1}(n)} \geq 1 - \frac{(t-n)^2}{2n} + \frac{(t-n)^3}{3n^2}.
\]

For \( n \geq 6 \), this inequality and Theorem 1 imply

\[ 1 - \theta(n) \geq \int_n^{n+2/3} \frac{\psi_{n-1}(t)}{\psi_{n-1}(n)} \, dt \geq \frac{2}{3} + \frac{4}{81n} + \frac{4}{243n^2}, \]

and the upper bound for \( \theta(n) \) follows. It can be verified that the upper bound still holds for \( 1 \leq n \leq 5 \). This completes the proof of Theorem 3. □

**Proof of Theorem 4.** Using the Taylor expansion of \( \psi_{n-1} \) at \( t = n \), we have

\[ 1 - \theta(n) = \sum_{k=0}^{N-1} \frac{\psi_{n-1}^{(k)}(n) \lambda(n)}{\psi_{n-1}(n)} (\lambda(n) - n)^{k+1} (k + 1)! + R_n. \]

We estimate the error term \( R_n \) as follows. Recall that

\[ \psi_{n-1}(t) = w_{N,n-1}(t) \psi_{n-1}(t)/t^N. \]
It is not difficult to show by induction that \(|\| w_{N,n-1} \| = \max\{|w_{N,n-1}(t)| : n \leq t \leq n + \log 2\} \leq N!n^{N/2}\). Therefore,

\[
|R_n| = \left| \int_n^{\lambda(n)} \frac{(t-n)^N}{N!} \frac{\psi_{n-1}(\xi(t))}{\psi_{n-1}(n)} \, dt \right| \leq \frac{(\lambda(n) - n)^{N+1}}{(N+1)!} \frac{\| w_{N,n-1} \|}{n^N} \\
\leq \frac{(\lambda(n) - n)^{N+1}}{(N+1)n^{N/2}} \to 0 \quad \text{as} \quad N \to \infty.
\]

Let \(a_k = \psi_{n-1}^{(k)}(n) / \psi_{n-1}(n)\) for \(k \geq 0\). It is easy to verify that \(a_0 = 1\), \(a_1 = 0\), and, for \(k \geq 2\), \(a_k = -(k-1)[a_{k-1} + a_{k-2}] / n\). Then we obtain

\[
1 - \theta(n) = \sum_{k=0}^{\infty} a_k \frac{(\lambda(n) - n)^{k+1}}{(k+1)!}.
\]

From this we can derive the asymptotic expansion of \(\lambda(n)\). This finishes the proof. \(\square\)

4. Remarks

It is not difficult to see that \(\lambda \text{ median}(\Gamma(\alpha, \lambda)) = \text{median}(\Gamma(\alpha, 1))\). Therefore, we have the following corollary from Theorem 1.

**Corollary 5.** For \(n \geq 1\) and \(\lambda > 0\), we have

\[
-\frac{1}{3\lambda} \leq \text{median}(\Gamma(n, \lambda)) - \text{mean}(\Gamma(n, \lambda)) \leq -\frac{1 + \log 2}{\lambda}.
\]

The bounds are best possible.

Using the fact that \(\chi_n^2 = \Gamma(\frac{n}{2}, \frac{1}{2})\), we have

**Corollary 6.** For \(m \geq 1\),

\[
-\frac{2}{3} \leq \text{median}(\chi_{2m}^2) - \text{mean}(\chi_{2m}^2) \leq 2(-1 + \log 2).
\]

The bounds are best possible.

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References


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