ON THE MÜNTZ RATIONAL APPROXIMATION RATE

S. P. ZHOU

(Communicated by J. Marshall Ash)

Abstract. The present paper constructs a counterexample to show that a conjecture of Newman concerning rational approximation rate of arbitrary Markov system is generally not true.

Let $C_{[0,b]}$ be the class of all real continuous functions $f$ on $[0, b]$. For $f \in C_{[0,b]}$,

$$
\omega(f, t) = \max \{|f(x + h) - f(x)| : x \in [0, b - h], \ 0 < h \leq t\},
$$

$$
\|f\|_{[0,b]} = \max_{x \in [0,b]} |f(x)|, \quad \text{and} \quad \|f\| = \|f\|_{[0,1]}.
$$

Given a subspace $S$ of $C_{[0,b]}$, let

$$
R(S) = \{P(x)/Q(x) : P(x) \in S, \ Q(x) \in S, \ Q(x) > 0, \ x \in (0, b)\},
$$

where we assume that $P(0)/Q(0) = \lim_{x \to 0^+} P(x)/Q(x)$ is finite in the case $Q(0) = 0$. For a sequence $\Lambda = \{\lambda_n\}_{n=0}^\infty$, write

$$
R(\Lambda) = R(\text{span}\{x^{\lambda_n}\}).
$$

From Müntz's theorem, it is well known that the linear combinations of $\{x^{\lambda_n}\}$ for

(1) \quad 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots

are dense if and only if $\sum_{n=1}^\infty 1/\lambda_n = \infty$.

Concerning the rational case, in 1976, Somorjai [5] showed a beautiful result that, under (1), $R(\Lambda)$ is always dense in $C_{[0,1]}$. In 1978, Bak and Newman [2] proved that, if $\{\lambda_n\}$ is any sequence of distinct positive real numbers, then $R(\Lambda)$ is dense in $C_{[0,1]}$, too. Our recent work [6] showed that $R(\Lambda)$ is always dense for any sequence of real numbers $\{\lambda_n\}$ with infinitely many distinct elements.
Write

\[ \Lambda_n = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \}, \]

\[ E_n = \{ e^{-\lambda_1 t}, e^{-\lambda_2 t}, \ldots, e^{-\lambda_n t} \}, \quad E = \{ e^{-\lambda_1 t}, e^{-\lambda_2 t}, \ldots \}, \]

\[ R(\Lambda_n) = R(\text{span}\{x^{\lambda_k} \mid \lambda_k \in \Lambda_n\}), \quad R(E_n) = R(\text{span}(E_n)), \]

\[ R_n(f, \Lambda) = \min_{\gamma \in R(\Lambda_n)} \|f - \gamma\| \quad \text{for } f \in C_{[0,1]}, \]

\[ R_n(f, E) = \min_{\gamma \in R(E_n)} \|f - \gamma\|_{[0,\infty)} \quad \text{for } f \in C_{[0,\infty]}. \]

On the quantitative Müntz rational approximation rate, Bak [1] proved that, if \( f \in C_{[0,1]} \) and \( \{ \lambda_n \} \) is a sequence with (1) and \( \lambda_k - \lambda_{k-1} \geq k \) for all \( k \geq 2 \), then

\[ R_n(f, \Lambda) \leq C \omega(f, n^{-1}), \]

where here and throughout the paper \( C \) always indicates an absolute constant which may have different values in different places, while \( C(x) \) indicates a positive constant only depending upon \( x \).

Newman [4] raised the following two problems on this topic (Newman said that even he did not believe Problem 10.4):  

**Problem 10.3.** Is it true that for any \( f \in C_{[0,1]} \) there exists \( R(x) \in R(\Lambda_n) \) such that

\[ \|f - R\| \leq C \omega(f, n^{-1})? \]

**Problem 10.4.** The same conclusion as the above problem holds where \( x^{\lambda_k} \) are replaced by \( \Psi_k(x) \) for any Markov system \( \{\Psi_k(x)\} \).

An infinite Markov system on an interval \([a, b]\) is a collection of continuous functions on \([a, b]\), \( \mathcal{M} := \{\Psi_1 = 1, \Psi_2, \Psi_3, \ldots\} \), with the property that, if an element of the real linear span of the first \( n \) vanishes at \( n \) points, then it vanishes identically.

The present paper will construct a counterexample to show Problem 10.4 is generally not true for a Markov system in the unbounded interval \([0,\infty)\).

2

We will prove the following result, which gives a negative answer to Problem 10.4 (for a Markov system in \([0,\infty)\)).

**Theorem.** Let \( \Lambda^* = \{\lambda_k\}_{k=1}^{\infty} \) and

\[ A = \{ A_{1,1}, A_{1,2}, A_{1,3}, A_{1,4}, A_{2,1}, A_{2,2}, A_{2,3}, A_{2,4}, \ldots, A_{1,2^n}, A_{2,2^n}, \ldots, A_{n-1,2^n}, A_n, A_{n,2^n}, \ldots, A_{n,2^n+1}, A_{1,2^n+2}, \ldots, A_n,2^n+1, \ldots, A_{n,2^n+1}, A_{1,2^n+1}, \ldots, A_{n,2^n+1}, \ldots, A_{n,2^n+1}, \ldots \} := \{\lambda_k\}_{k=1}^{\infty}, \]

where

\[ A_{i,j} = \lambda_i^* + j - 1, \quad i, j = 1, 2, \ldots, \quad \lambda_i^* = \begin{cases} 0, & i = 1, \\ i^{-2}, & i \geq 2. \end{cases} \]
Furthermore, let \( \{s_n\} \) be an increasing sequence with the following properties:

\[
\lim_{n \to \infty} s_n = +\infty, \quad s_n \sim s_{2n}, \quad \lim_{n \to \infty} (s_n/n) = 0.
\]

Then \( E \) is a Markov system on \([0, \infty)\) and there exists a function \( f \in C_{[0, \infty]} \) such that

\[
\lim_{n \to \infty} \frac{R_n(f, E)}{s_n o(f, n^{-1})} > 0.
\]

**Proof.** It is a very clear fact that \( E \) is a Markov system since \( \lambda_k, \; k = 1, 2, \ldots, \) are distinct. Let \( T_n(t) := T_n(t, \Lambda^*) \) be the generalized Chebyshev polynomial of degree \( n \) associated with the Markov system \( \{e^{-\lambda_1 t}, e^{-\lambda_2 t}, \ldots\} \) on \([0, \infty)\), that is, the linear form

\[
T_n(t) = C_0 \left( e^{-\lambda_1 t} - \sum_{k=1}^{n-1} C_k e^{-\lambda_k t} \right),
\]

where \( C_k, \; k = 1, 2, \ldots, n - 1, \) are chosen so that \( \sum_{k=1}^{n-1} C_k e^{-\lambda_k t} \) is the best approximant to \( e^{-\lambda_1 t} \) from \( \text{span}\{e^{-\lambda_1 t}\}^{n-1} \) and where \( C_0 \) is chosen so that \( \|T_n\|_{[0, \infty)} = 1 \).

By the well-known results in approximation theory, there exists an ordinary polynomial \( Q_n(x) \) with sufficiently large degree \( m_n \geq 1 \) such that

\[
\|T_n(t) - Q_n(e^{-t})\|_{[0, \infty)} \leq n^{-1}.
\]

Now we may select a sequence \( \{n_l\} \) by induction. Let \( n_1 = 2 \). Suppose \( n_l \) is given. Let

\[
M_l := 2^n_l + 2.
\]

Choose \( n_{l+1} \) satisfying the following properties:

\[
n_{l+1} \geq \lceil \log_2 M_l \rceil + 1,
\]

\[
e^{l+1} := \sqrt{\frac{3 M_{l+1}}{M_{l+1}}} \leq \min \left\{ \frac{e_l s_{M_l}}{4}, \sum_{k=1}^{l} e_k \left\| \frac{d}{dt} Q_k(e^{-t}) \right\|_{[0, \infty)} \right\}.
\]

By (2), (7) is possible. Define

\[
F_l(t) = \sum_{k=1}^{l} e_k Q_{M_k}(e^{-t}), \quad f(t) = \sum_{k=1}^{\infty} e_k Q_{M_k}(e^{-t}).
\]

It is quite clear that \( f \in C_{[0, \infty]} \) follows from (3) and (7). For any rational \( r(t) \in R(E_{M_l^*}) \),

\[
\|f(t) - r(t)\|_{[0, \infty)} \geq \|F_l(t) - r(t)\|_{[0, \infty)} - 2 \sum_{k=l+1}^{\infty} e_k.
\]

---

\( ^2 \) By \( A_n \sim B_n \), we mean that there exists a positive constant \( M \) independent of \( n \) such that \( M^{-1} \leq A_n/B_n \leq M \).
We have the estimate

\[ (9) \quad \| F_t - r \|_{[0, \infty)} \geq \varepsilon_t (1 - M_t^{-1}). \]

In fact, that (9) fails will lead to a contradiction. Because of the definition, \( T_{M_t}(t) \) oscillates between \( \pm 1 \) exactly \( M_t \) times on \([0, \infty)\).\(^3\) Assume, for \( 0 \leq x_1 < x_2 < \cdots < x_{M_t} \leq \infty \),

\[ T_{M_t}(x_j) = \varepsilon(-1)^j, \quad \varepsilon = \pm 1. \]

From (3),

\[ \text{sgn}(Q_{M_t}(x_j)) = \text{sgn}(T_{M_t}(x_j)), \quad |Q_{M_t}(x_j)| \geq 1 - M_t^{-1}. \]

Suppose (9) fails. Then

\[ \text{sgn}((r - F_{t-1})(x_j)) = \text{sgn}((\varepsilon_t Q_{M_t} - F_t + r)(x_j)) = \text{sgn}(Q_{M_t}(x_j)), \]

which means \( r - F_{t-1} \) vanishes at least \( M_t - 1 \) times on \([0, \infty)\). It is impossible since \( r - F_{t-1} \in R(E_{M_t-2}) \) by (4)-(6) and by some direct arguments.

Therefore, combining (7)-(9) yields that

\[ \| f(t) - r(t) \|_{[0, \infty)} \geq \varepsilon_t (1 - M_t^{-1}) - \varepsilon_t s_{M_t}^{-1}, \]

so that

\[ (10) \quad R_{M_t}^\bullet(f, E) \geq \varepsilon_t (1 - M_t^{-1}) - \varepsilon_t s_{M_t}^{-1}. \]

On the other hand,

\[ \omega(f, (M_t^\bullet)^{-1}) \leq 3M_t^{-1} \sum_{k=1}^{t-1} \varepsilon_k \left\| \frac{d}{dt} Q_{M_t} (e^{-t}) \right\|_{[0, \infty)} + 3 \varepsilon_t \omega(Q_{M_t}(e^{-t}), M_t^{-1}) + 4 \sum_{k=t+1}^{\infty} \varepsilon_k := \Sigma_1 + \Sigma_2 + \Sigma_3. \]

It follows from (7) that

\[ \Sigma_3 \leq 2 \varepsilon_t s_{M_t}^{-1} \text{ and } \Sigma_1 \leq 3 \varepsilon_t s_{M_t}^{-1}. \]

Applying (2) and an inequality for derivatives of generalized Müntz polynomials of Newman [3] we have

\[ \Sigma_2 \leq 3 \| Q_{M_t}(e^{-1}) - T_{M_t}(t) \|_{[0, \infty)} + 3M_t^{-1} \| T_{M_t}' \|_{[0, \infty)} = O(M_t^{-1}) \]

since \( \sum_{k=2}^{\infty} \lambda_k^* < +\infty \). Putting together the above estimates and (10) we then have

\[ \frac{R_{M_t}^\bullet(f, E)}{s_{M_t}^\bullet \omega(f, (M_t^\bullet)^{-1})} \geq \frac{C R_{M_t}^\bullet(f, E)}{s_{M_t} \omega(f, (M_t^\bullet)^{-1})} > C. \]

The theorem is completed. \( \Box \)

\(^3\) We may take \( \infty \) as an alternating point.
THE MÜNTZ RATIONAL APPROXIMATION RATE

REFERENCES

2. J. Bak and D. J. Newman, Rational combinations of $x^{\lambda_k}$, $\lambda_k \geq 0$, are always dense in $C[0,1]$, J. Approx. Theory 23 (1978), 155–157.

Department of Mathematics, Statistics, and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5
Current address: Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1
E-mail address: zhou@approx.math.ualberta.ca