ON THE MÜNTZ RATIONAL APPROXIMATION RATE

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Abstract. The present paper constructs a counterexample to show that a conjecture of Newman concerning rational approximation rate of arbitrary Markov system is generally not true.

Let \( C_{[0,b]} \) be the class of all real continuous functions \( f \) on \([0, b]\). For \( f \in C_{[0,b]} \),
\[
\omega(f, t) = \max \{|f(x + h) - f(x)| : x \in [0, b - h], \ 0 < h \leq t\},
\]
\[
\|f\|_{[0,b]} = \max_{x \in [0,b]} |f(x)|, \quad \text{and} \quad \|f\| = \|f\|_{[0,1]}.
\]
Given a subspace \( S \) of \( C_{[0,b]} \), let
\[
R(S) = \{P(x)/Q(x) : P(x) \in S, \ Q(x) \in S, \ Q(x) > 0, \ x \in (0, b]\},
\]
where we assume that \( P(0)/Q(0) = \lim_{x \to 0^+} P(x)/Q(x) \) is finite in the case \( Q(0) = 0 \). For a sequence \( \Lambda = \{\lambda_n\}_{n=0}^{\infty} \), write
\[
R(\Lambda) = R(\text{span}\{x^{\lambda_n}\}).
\]

From Müntz's theorem, it is well known that the linear combinations of \( \{x^{\lambda_n}\} \) for
\[
0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots
\]
are dense if and only if \( \sum_{n=1}^{\infty} 1/\lambda_n = \infty \).

Concerning the rational case, in 1976, Somorjai [5] showed a beautiful result that, under (1), \( R(\Lambda) \) is always dense in \( C_{[0,1]} \). In 1978, Bak and Newman [2] proved that, if \( \{\lambda_n\} \) is any sequence of distinct positive real numbers, then \( R(\Lambda) \) is dense in \( C_{[0,1]} \), too. Our recent work [6] showed that \( R(\Lambda) \) is always dense for any sequence of real numbers \( \{\lambda_n\} \) with infinitely many distinct elements.

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\(^1\)For \( b = \infty \), we assume further that \( \lim_{x \to \infty} f(x) \) exists and is finite.
Write

\[ \Lambda_n = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \}, \]

\[ E_n = \{ e^{-\lambda_1 t}, e^{-\lambda_2 t}, \ldots, e^{-\lambda_n t} \}, \]

\[ R(\Lambda_n) = R(\text{span}\{x^{\lambda_k} : \lambda_k \in \Lambda_n\}), \quad R(E_n) = R(\text{span}(E_n)), \]

\[ R_n(f, \Lambda) = \min_{y \in R(\Lambda_n)} \| f - y \| \quad \text{for } f \in C_{[0,1]}, \]

\[ R_n(f, E) = \min_{y \in R(E_n)} \| f - y \|_{[0,\infty)} \quad \text{for } f \in C_{[0,\infty]} . \]

On the quantitative Müntz rational approximation rate, Bak [1] proved that, if \( f \in C_{[0,1]} \) and \( \{ \lambda_n \} \) is a sequence with (1) and \( \lambda_k - \lambda_{k-1} \geq k \) for all \( k \geq 2 \), then

\[ R_n(f, \Lambda) \leq C \omega(f, n^{-1}), \]

where here and throughout the paper \( C \) always indicates an absolute constant which may have different values in different places, while \( C(x) \) indicates a positive constant only depending upon \( x \).

Newman [4] raised the following two problems on this topic (Newman said that even he did not believe Problem 10.4):

**Problem 10.3.** Is it true that for any \( f \in C_{[0,1]} \) there exists \( R(x) \in R(\Lambda_n) \) such that

\[ \| f - R \| \leq C \omega(f, n^{-1})? \]

**Problem 10.4.** The same conclusion as the above problem holds where \( x^{\lambda_k} \) are replaced by \( \Psi_k(x) \) for any Markov system \( \{\Psi_k(x)\} \).

An infinite Markov system on an interval \([a, b]\) is a collection of continuous functions on \([a, b]\), \( \mathcal{A} := \{ \Psi_1 = 1, \Psi_2, \Psi_3, \ldots \} \), with the property that, if an element of the real linear span of the first \( n \) vanishes at \( n \) points, then it vanishes identically.

The present paper will construct a counterexample to show Problem 10.4 is generally not true for a Markov system in the unbounded interval \([0, \infty)\).

We will prove the following result, which gives a negative answer to Problem 10.4 (for a Markov system in \([0, \infty)\)).

**Theorem.** Let \( \Lambda^* = \{ \lambda_k^* \}_{k=1}^{\infty} \) and

\[ A = \{ A_1,1, A_1,2, A_1,3, A_1,4, A_2,1, A_2,2, A_2,3, A_2,4, \ldots, A_1,2^n, \]

\[ A_2,2^n, \ldots, A_n,1, A_n,2, \ldots, A_n,2^n, A_1,2^n+1, \ldots, A_n,2^n+1, \]

\[ A_1,2^n+2, \ldots, A_n,2^n+2, \ldots, A_1,2^{n+1}-1, \ldots, A_n,2^{n+1}-1 \} := \{ \lambda_k \}_{k=1}^{\infty}, \]

where

\[ A_{i,j} = \lambda_i^* + j - 1, \quad i, j = 1, 2, \ldots, \quad \lambda_i^* = \begin{cases} 0, & i = 1, \\ i^{-2}, & i \geq 2. \end{cases} \]
Furthermore, let \( \{s_n\} \) be an increasing sequence with the following properties:

\[
\lim_{n \to \infty} s_n = +\infty, \quad s_n \sim s_{2n}, \quad \lim_{n \to \infty} \left( \frac{s_n}{n} \right) = 0.
\]

Then \( E \) is a Markov system on \( [0, \infty) \) and there exists a function \( f \in C_{[0, \infty)} \) such that

\[
\lim_{n \to \infty} \sup_{E_n} \frac{R_n(f, E)}{s_n} > 0.
\]

**Proof.** It is a very clear fact that \( E \) is a Markov system since \( \lambda_k, \ k = 1, 2, \ldots, \) are distinct. Let \( T_n(t) := T_n(t, \lambda^*) \) be the generalized Chebyshev polynomial of degree \( n \) associated with the Markov system \( \{e^{-\lambda_k^* t}, e^{-\lambda_k^* t}, \ldots\} \) on \( [0, \infty) \), that is, the linear form

\[
T_n(t) = C_0 \left( e^{-\lambda_1^* t} - \sum_{k=1}^{n-1} C_k e^{-\lambda_k^* t} \right),
\]

where \( C_k, \ k = 1, 2, \ldots, n-1, \) are chosen so that \( \sum_{k=1}^{n-1} C_k e^{-\lambda_k^* t} \) is the best approximant to \( e^{-\lambda_1^* t} \) from \( \text{span}\{e^{-\lambda_k^* t}\}_{k=1}^{n-1} \) and where \( C_0 \) is chosen so that \( \|T_n\|_{[0, \infty)} = 1 \).

By the well-known results in approximation theory, there exists an ordinary polynomial \( Q_n(x) \) with sufficiently large degree \( m_n \geq 1 \) such that

\[
\|T_n(t) - Q_n(e^{-t})\|_{[0, \infty)} \leq n^{-1}.
\]

Now we may select a sequence \( \{n_l\} \) by induction. Let \( n_1 = 2 \). Suppose \( n_l \) is given. Let

\[
M_l^* := n_l 2^n, \quad M_l := 2M_l^* + 2.
\]

Choose \( n_{l+1} \) satisfying the following properties:

\[
n_{l+1} \geq \lceil \log_2 (m_{M_l^*} + 1) \rceil, \quad \varepsilon_{l+1} := \sqrt{\frac{3M_{M_l^*}}{M_{l+1}}} \leq \min \left\{ \frac{\varepsilon_l M_l^*}{4}, \sum_{k=1}^{l} \varepsilon_k \left\| \frac{d}{dt} Q_l(e^{-t}) \right\|_{[0, \infty)} \right\}.
\]

By (2), (7) is possible. Define

\[
F_l(t) = \sum_{k=1}^{l} \varepsilon_k Q_{M_k}(e^{-t}), \quad f(t) = \sum_{k=1}^{\infty} \varepsilon_k Q_{M_k}(e^{-t}).
\]

It is quite clear that \( f \in C_{[0, \infty)} \) follows from (3) and (7). For any rational \( r(t) \in R(E_{M_l^*}) \),

\[
\|f(t) - r(t)\|_{[0, \infty)} \geq \|F_l(t) - r(t)\|_{[0, \infty)} - 2 \sum_{k=l+1}^{\infty} \varepsilon_k.
\]

\[2 \] By \( A_n \sim B_n \), we mean that there exists a positive constant \( M \) independent of \( n \) such that \( M^{-1} \leq A_n/B_n \leq M \).
We have the estimate

\[ \|F_1 - r\|_{[0, \infty)} \geq \varepsilon_i(1 - M_i^{-1}). \]

In fact, that (9) fails will lead to a contradiction. Because of the definition, \( T_{M_i}(t) \) oscillates between \( \pm 1 \) exactly \( M_i \) times on \([0, \infty)\). Assume, for \( 0 \leq x_1 < x_2 < \cdots < x_{M_i} \leq \infty \),

\[ T_{M_i}(x_j) = \varepsilon(-1)^j, \quad \varepsilon = \pm 1. \]

From (3),

\[ \text{sgn}(Q_{M_i}(x_j)) = \text{sgn}(T_{M_i}(x_j)), \quad |Q_{M_i}(x_j)| \geq 1 - M_i^{-1}. \]

Suppose (9) fails. Then

\[ \text{sgn}((r - F_{i-1})(x_j)) = \text{sgn}((\varepsilon_iQ_{M_i} - F_i + r)(x_j)) = \text{sgn}(Q_{M_i}(x_j)), \]

which means \( r - F_{i-1} \) vanishes at least \( M_i - 1 \) times on \([0, \infty)\). It is impossible since \( r - F_{i-1} \in R(E_{M_i-2}) \) by (4)-(6) and by some direct arguments. Therefore, combining (7)-(9) yields that

\[ \|f(t) - r(t)\|_{[0, \infty)} \geq \varepsilon_i(1 - M_i^{-1}) - \varepsilon_i s_{M_i}^{-1}, \]

so that

\[ R_{M_i^*}(f, E) \geq \varepsilon_i(1 - M_i^{-1}) - \varepsilon_i s_{M_i}^{-1}. \]

On the other hand,

\[ \omega(f, (M_i^*)^{-1}) \leq 3M_i^{-1} \sum_{k=1}^{l-1} \varepsilon_k \left\| \frac{d}{dt}Q_{M_i}(e^{-t}) \right\|_{[0, \infty)} + 3\varepsilon_i \omega(Q_{M_i}(e^{-t}), M_i^{-1}) \]

\[ + 4 \sum_{k=l+1}^{\infty} \varepsilon_k := \Sigma_1 + \Sigma_2 + \Sigma_3. \]

It follows from (7) that

\[ \Sigma_3 \leq 2\varepsilon_i s_{M_i}^{-1} \quad \text{and} \quad \Sigma_1 \leq 3\varepsilon_i s_{M_i}^{-1}. \]

Applying (2) and an inequality for derivatives of generalized Müntz polynomials of Newman [3] we have

\[ \Sigma_2 \leq 3\|Q_{M_i}(e^{-t}) - T_{M_i}(t)\|_{[0, \infty)} + 3M_i^{-1}\|T_{M_i}'\|_{[0, \infty)} = O(M_i^{-1}) \]

since \( \sum_{k=2}^{\infty} \lambda_k^* < +\infty \). Putting together the above estimates and (10) we then have

\[ \frac{R_{M_i^*}(f, E)}{s_{M_i^*} \omega(f, (M_i^*)^{-1})} \geq \frac{C R_{M_i^*}(f, E)}{s_{M_i} \omega(f, (M_i^*)^{-1})} > C. \]

The theorem is completed. \( \square \)

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3 We may take \( \infty \) as an alternating point.
REFERENCES


2. J. Bak and D. J. Newman, *Rational combinations of $x^{\lambda_k}$, $\lambda_k \geq 0$, are always dense in $C_{[0,1]}$*, J. Approx. Theory 23 (1978), 155–157.


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