ON THE MÜNTZ RATIONAL APPROXIMATION RATE

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ABSTRACT. The present paper constructs a counterexample to show that a conjecture of Newman concerning rational approximation rate of arbitrary Markov system is generally not true.

Let \( C_{[0, b]} \) be the class of all real continuous functions \( f \) on \([0, b]\).\(^1\) For \( f \in C_{[0, b]} \),

\[
\omega(f, t) = \max\{|f(x + h) - f(x)|: x \in [0, b - h], \ 0 < h \leq t\},
\]

\[
\|f\|_{[0, b]} = \max_{x \in [0, b]} |f(x)|,
\]

and \( \|f\| = \|f\|_{[0,1]} \).

Given a subspace \( S \) of \( C_{[0, b]} \), let

\[
R(S) = \{P(x)/Q(x): P(x) \in S, \ Q(x) \in S, \ Q(x) > 0, \ x \in (0, b]\},
\]

where we assume that \( P(0)/Q(0) = \lim_{x \to 0^+} P(x)/Q(x) \) is finite in the case \( Q(0) = 0 \). For a sequence \( \Lambda = \{\lambda_n\}_{n=0}^{\infty} \), write

\[
R(\Lambda) = R(\text{span}\{x^{\lambda_n}\}).
\]

From Müntz's theorem, it is well known that the linear combinations of \( \{x^{\lambda_n}\} \) for

\[
0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots
\]

are dense if and only if \( \sum_{n=1}^{\infty} 1/\lambda_n = \infty \).

Concerning the rational case, in 1976, Somorjai [5] showed a beautiful result that, under (1), \( R(\Lambda) \) is always dense in \( C_{[0, 1]} \). In 1978, Bak and Newman [2] proved that, if \( \{\lambda_n\} \) is any sequence of distinct positive real numbers, then \( R(\Lambda) \) is dense in \( C_{[0, 1]} \), too. Our recent work [6] showed that \( R(\Lambda) \) is always dense for any sequence of real numbers \( \{\lambda_n\} \) with infinitely many distinct elements.
Write
\[
\Lambda_n = \{\lambda_1, \lambda_2, \ldots, \lambda_n\},
\]
\[
E_n = \{e^{-\lambda_1t}, e^{-\lambda_2t}, \ldots, e^{-\lambda_nt}\}, \quad E = \{e^{-\lambda_1t}, e^{-\lambda_2t}, \ldots\},
\]
\[
R(\Lambda_n) = R(\text{span}\{x^{\lambda_k}: \lambda_k \in \Lambda_n\}), \quad R(E_n) = R(\text{span}(E_n)),
\]
\[
R_n(f, \Lambda) = \min_{\gamma \in R(\Lambda_n)} \|f - \gamma\| \quad \text{for} \ f \in C_{[0,1]},
\]
\[
R_n(f, E) = \min_{\gamma \in R(E_n)} \|f - \gamma\|_{[0,\infty)} \quad \text{for} \ f \in C_{[0,\infty]}.
\]

On the quantitative Müntz rational approximation rate, Bak [1] proved that, if \( f \in C_{[0,1]} \) and \( \{\lambda_n\} \) is a sequence with (1) and \( \lambda_k - \lambda_{k-1} \geq k \) for all \( k \geq 2 \), then
\[
R_n(f, \Lambda) \leq C \omega(f, n^{-1}),
\]
where here and throughout the paper \( C \) always indicates an absolute constant which may have different values in different places, while \( C(x) \) indicates a positive constant only depending upon \( x \).

Newman [4] raised the following two problems on this topic (Newman said that even he did not believe Problem 10.4):

**Problem 10.3.** Is it true that for any \( f \in C_{[0,1]} \) there exists \( R(x) \in R(\Lambda_n) \) such that
\[
\|f - R\| \leq C \omega(f, n^{-1}).
\]

**Problem 10.4.** The same conclusion as the above problem holds where \( x^{\lambda_k} \) are replaced by \( \Psi_k(x) \) for any Markov system \( \{\Psi_k(x)\} \).

An infinite Markov system on an interval \([a, b]\) is a collection of continuous functions on \([a, b]\), \( \mathcal{M} := \{\Psi_1 = 1, \Psi_2, \Psi_3, \ldots\} \), with the property that, if an element of the real linear span of the first \( n \) vanishes at \( n \) points, then it vanishes identically.

The present paper will construct a counterexample to show Problem 10.4 is generally not true for a Markov system in the unbounded interval \([0, \infty)\).

We will prove the following result, which gives a negative answer to Problem 10.4 (for a Markov system in \([0, \infty)\)).

**Theorem.** Let \( \Lambda^* = \{\lambda_k^*\}_{k=1}^\infty \) and
\[
A = \{A_{1,1}, A_{1,2}, A_{1,3}, A_{1,4}, A_{2,1}, A_{2,2}, A_{2,3}, A_{2,4}, \ldots, A_{1,2^n}, A_{2,2^n}, \ldots, A_{n-1,2^n}, A_{n,1}, A_{n,2}, \ldots, A_{n,2^n}, A_{1,2^{n+1}}, \ldots, A_{n,2^{n+1}}, A_{1,2^{n+2}}, \ldots, A_{n,2^{n+2}}, \ldots, A_{1,2^{n+1}}, \ldots, A_{n,2^{n+1}}, \ldots\} := \{\lambda_k^*\}_{k=1}^\infty,
\]
where
\[
A_{i,j} = \lambda_i^* + j + 1, \quad i, j = 1, 2, \ldots, \quad \lambda_i^* = \begin{cases} 0, & i = 1, \\ i^{-2}, & i \geq 2. \end{cases}
\]
Furthermore, let \( \{s_n\} \) be an increasing sequence with the following properties:

\[ \lim_{n \to \infty} s_n = +\infty, \quad s_{n+1} \sim s_n, \quad \lim_{n \to \infty} \frac{s_n}{n} = 0. \]

Then \( E \) is a Markov system on \([0, \infty)\) and there exists a function \( f \in C[0, \infty) \) such that

\[ \limsup_{n \to \infty} \frac{R_n(f, E)}{s_n \alpha(f, n^{-1})} > 0. \]

**Proof.** It is a very clear fact that \( E \) is a Markov system since \( \lambda_k, \, k = 1, 2, \ldots, \) are distinct. Let \( T_n(t) := T_n(t, A^*) \) be the generalized Chebyshev polynomial of degree \( n \) associated with the Markov system \( \{e^{-\lambda_1 t}, e^{-\lambda_2 t}, \ldots\} \) on \([0, \infty)\), that is, the linear form

\[ T_n(t) = C_0 \left( e^{-\lambda_1 t} - \sum_{k=1}^{n-1} C_k e^{-\lambda_k t} \right), \]

where \( C_k, \, k = 1, 2, \ldots, n - 1, \) are chosen so that \( \sum_{k=1}^{n-1} C_k e^{-\lambda_k t} \) is the best approximant to \( e^{-\lambda_1 t} \) from \( \text{span}\{e^{-\lambda_1 t}, e^{-\lambda_2 t}, \ldots\} \) and where \( C_0 \) is chosen so that \( \|T_n\|_{[0, \infty)} = 1 \).

By the well-known results in approximation theory, there exists an ordinary polynomial \( Q_n(x) \) with sufficiently large degree \( m_n \geq 1 \) such that

\[ \|T_n(t) - Q_n(e^{-t})\|_{[0, \infty)} \leq n^{-1}. \]

Now we may select a sequence \( \{n_l\} \) by induction. Let \( n_1 = 2 \). Suppose \( n_l \) is given. Let

\[ M_l := n_l 2^{n_l}, \]

so that \( M_l \) is a Markov system.

Choose \( n_{l+1} \) satisfying the following properties:

\[ n_{l+1} \geq [\log_2 m_{M_l} + 1], \]

\[ \varepsilon_{l+1} := \sqrt{\frac{M_{M_{l+1}}}{M_{l+1}}}, \quad \sum_{k=1}^{M_{M_{l+1}}} \varepsilon_k \left\| \frac{d}{dt} Q_i(e^{-t}) \right\|_{[0, \infty)} \leq \sqrt{\frac{M_{M_{M_{l+1}}}}{M_{l+1}}}, \]

By (2), (7) is possible. Define

\[ F_l(t) = \sum_{k=1}^{M_{M_{l+1}}} \varepsilon_k Q_{M_k}(e^{-t}), \quad f(t) = \sum_{k=1}^{M_{M_{l+1}}} \varepsilon_k Q_{M_k}(e^{-t}). \]

It is quite clear that \( f \in C[0, \infty) \) follows from (3) and (7). For any rational \( r(t) \in R(E_{M_l^*}) \),

\[ \|f(t) - r(t)\|_{[0, \infty)} \geq \|F_l(t) - r(t)\|_{[0, \infty)} - \sum_{k=l+1}^{M_{M_{l+1}}} \varepsilon_k. \]
We have the estimate

(9) \[ \| F_l - r \|_{(0, \infty)} \geq \varepsilon_l (1 - M_l^{-1}). \]

In fact, that (9) fails will lead to a contradiction. Because of the definition, \( T_{M_l}(t) \) oscillates between \( \pm 1 \) exactly \( M_l \) times on \([0, \infty)\). Assume, for \( 0 \leq x_1 < x_2 < \cdots < x_{M_l} \leq \infty \),
\[ T_{M_l}(x_j) = \varepsilon (-1)^j, \quad \varepsilon = \pm 1. \]

From (3),
\[ \text{sgn}(Q_{M_l}(x_j)) = \text{sgn}(T_{M_l}(x_j)), \quad |Q_{M_l}(x_j)| \geq 1 - M_l^{-1}. \]

Suppose (9) fails. Then
\[ \text{sgn}((r - F_{l-1})(x_j)) = \text{sgn}((\varepsilon_l Q_{M_l} - F_l + r)(x_j)) = \text{sgn}(Q_{M_l}(x_j)), \]
which means \( r - F_{l-1} \) vanishes at least \( M_l - 1 \) times on \([0, \infty)\). It is impossible since \( r - F_{l-1} \in R(E_{M_l-2}) \) by (4)-(6) and by some direct arguments.

Therefore, combining (7)-(9) yields that
\[ \| f(t) - r(t) \|_{(0, \infty)} \geq \varepsilon_l (1 - M_l^{-1}) - \varepsilon_l s_{M_l}^{-1}, \]
so that
\[ (10) \quad R_{M_l}(f, E) \geq \varepsilon_l (1 - M_l^{-1}) - \varepsilon_l s_{M_l}^{-1}. \]

On the other hand,
\[ \omega(f, (M_l^*)^{-1}) \leq 3M_l^{-1} \sum_{k=1}^{l-1} \varepsilon_k \left\| \frac{d}{dt}Q_{M_k}(e^{-t}) \right\|_{(0, \infty)} + 3\varepsilon_l \omega(Q_{M_l}(e^{-t}), M_l^{-1}) \]
\[ + 4 \sum_{k=l+1}^{\infty} \varepsilon_k := \Sigma_1 + \Sigma_2 + \Sigma_3. \]

It follows from (7) that
\[ \Sigma_3 \leq 2\varepsilon_l s_{M_l}^{-1} \quad \text{and} \quad \Sigma_1 \leq 3\varepsilon_l s_{M_l}^{-1}. \]

Applying (2) and an inequality for derivatives of generalized Müntz polynomials of Newman [3] we have
\[ \Sigma_2 \leq 3\|Q_{M_l}(e^{-1}) - T_{M_l}(t)\|_{(0, \infty)} + 3M_l^{-1}\|T_{M_l}'\|_{(0, \infty)} = O(M_l^{-1}) \]
since \( \sum_{k=2}^{\infty} \lambda_k^* < +\infty \). Putting together the above estimates and (10) we then have
\[ \frac{R_{M_l}(f, E)}{s_{M_l}(f, (M_l^*)^{-1})} \geq \frac{C R_{M_l}(f, E)}{s_{M_l}(f, (M_l^*)^{-1})} > C. \]

The theorem is completed. □
REFERENCES


2. J. Bak and D. J. Newman, *Rational combinations of $x^{\lambda_k}$, $\lambda_k \geq 0$, are always dense in $C_{[0,1]}$*, J. Approx. Theory 23 (1978), 155–157.


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