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UNION THEOREM FOR COHOMOLOGICAL DIMENSION: A SIMPLE COUNTEREXAMPLE

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ABSTRACT. An elementary counterexample to the Union Theorem for cohomological dimension with coefficients in $\mathbf{Z}/2^\infty$ is presented.

Recently, Dranishnikov, Repovš, and Ščepin [DRS] have found a subset A of R^4 such that $\dim_{\mathbf{Q}/\mathbf{Z}} A = \dim_{\mathbf{Q}/\mathbf{Z}}(R^4 - A) = 1$ which demonstrates that the Union Theorem ($\dim_G(A \cup B) \leq \dim_G A + \dim_G B + 1$) is not valid for $G = \mathbf{Q}/\mathbf{Z}$. Previously, partial validity of the Union Theorem had been established by Rubin [Ru] ($G = \mathbf{Z}$) and Dydak and Walsh [DW] ($G = \mathbf{Z}/p$, G a subring of $\mathbf{Q}$).

The reasoning in [DRS] is quite ingenious and uses recent results of the authors concerning embedding problems for cohomological dimension. The purpose of this note is to present a simple construction of a subset A of S^4 such that $\dim_{\mathbf{Z}/2} A = 1$ and $\dim_G(S^4 - A) = 1$ for any abelian group G divisible by 2. Taking $G = \mathbf{Z}/2^\infty$ (i.e., the direct limit of $\mathbf{Z}/2 \to \mathbf{Z}/4 \to \cdots \to \mathbf{Z}/2^k \to \cdots$) one obtains a counterexample to the Union Theorem for cohomological dimension. Indeed, Bockstein Inequalities [Ku] imply that $\dim_{\mathbf{Z}/2^\infty} A = 1$.

Basic Lemma. *Given a compact PL submanifold M^4 of S^4 there is a compactum $X \subset S^4 - M$ such that $\dim_{\mathbf{Z}/2} X = 1$ and the inclusion induced homomorphism*

$$H^1(S^4 - X; G) \to H^1(M; G)$$

is an epimorphism for any abelian group G divisible by 2.

Proof. Enlarge M^4 to $N^4 \sim M^4$. Let A_1 be the 2-skeleton of $S^4 - \operatorname{int} N^4$. One can define inductively a sequence of 2-polyhedra A_i and their regular neighborhoods X_i in $S^4 - \operatorname{int} N^4$ as follows:

1. Choose a triangulation L_i of X_i of mesh at most $1/i$.
2. Given a 2-simplex $\Delta \subset A_i$ of L_i, find a real projective plane $P(\Delta)$ contained in a 4-simplex Δ' of L_i so that $\partial\Delta$ represents the generator of the fundamental group of $P(\Delta)$ and $P(\Delta) - \partial\Delta \subset \operatorname{int}\Delta'$. We may assume that

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$(P(\Delta_1) - \partial\Delta_1) \cap (P(\Delta_2) - \partial\Delta_2) = \varnothing$ if $\Delta_1 \neq \Delta_2$ (strictly speaking, all we need is that $(P(\Delta_1) - \partial\Delta_1) \cap (P(\Delta_2) - \partial\Delta_2)$ is at most 0-dimensional).

3. Obtain A_{i+1} from A_i by removing the interior of Δ and adding $P(\Delta)$ for each 2-simplex of A_i.

4. X_{i+1} is a regular neighborhood of A_{i+1} in X_i and is contained in the $(1/i)$-neighborhood of A_{i+1}.

Notice that the inclusion-induced homomorphism

$$H_2(X_{i+1}; G) \to H_2(X_i; G)$$

is an epimorphism for any group G divisible by 2. Indeed, if $\sum c_i\Delta_i$ is a 2-cycle in A_i, we find $d_i \in G$ with $2d_i = c_i$. Since $c_i\Delta_i - d_i m_i$ (here m_i represents the chain in the projective plane $P(\Delta_i)$ obtained by adding all its 2-simplices so that $\partial m_i = 2\partial\Delta_i$) is a 2-cycle in a contractible set $\Delta_i' \subset X_i$, it is a boundary in X_i and $\sum c_i\Delta_i$ is homologous to $\sum d_i m_i$.

Define X as the intersection $\bigcap X_i$. Suppose $\alpha \in H^1(M; G) = H^2(S^4, M)$. By Lefschetz Duality (see [Sp, p. 297])

$$H^2(S^4, M) \approx H_2(S^4 - M; G),$$

so α is represented by an element β of $H_2(S^4 - M; G)$. Now, inductively, find $\beta_i \in H_2(X_i; G)$ so that $\beta = \beta_1$ in $H_2(S^4 - M; G)$ and $\beta_{i+1} = \beta_i$ in $H_2(X_i; G)$. Dually, one gets elements $\alpha_i \in H^1(S^4 - X_i; G)$ which pasted together form an extension of α to $S^4 - X$.

Suppose C is a closed subset of X and $\alpha : C \to RP^\infty$ represents an element of $H^1(C; \mathbf{Z}/2)$. There is an i and an extension $\tilde{\alpha} : Y_i \to RP^\infty$ of α to a subcomplex Y_i of X_i. We may assume that Y_i contains the 1-skeleton of X_i. Given a 2-simplex Δ of A_i one can extend $\tilde{\alpha}|\partial\Delta$ over $P(\Delta)$. Thus, α is extendible over $Y_i \cup A_{i+1}$. Extend α over a regular neighborhood of $Y_i \cup A_{i+1}$ containing X_{i+1}. Thus, α extends over X which proves that $\dim_{\mathbf{Z}/2} X \leq 1$. □

Theorem. *There is a subset* A *of* S^4 *such that* $\dim_{\mathbf{Z}/2} A = 1$ *and* $\dim_G(S^4 - A) = 1$ *for any abelian group* G *divisible by* 2.

Proof. Choose a countable family $\{M_i\}_{i\geq 1}$ of compact submanifolds of S^4 so that for any compact subset C of S^4 and any open set U containing C there is i with $C \subset M_i \subset U$. Choose $X_i \subset S^4 - M_i$ such that $\dim_{\mathbf{Z}/2} X_i \leq 1$ and

$$H^1(S^4 - X_i; G) \to H^1(M_i; G)$$

is an epimorphism for any abelian group divisible by 2. Let $A \supset \bigcup X_i$ be a G_δ-set so that $\dim_{\mathbf{Z}/2} A \leq 1$. Given a compact set C in $S^4 - A$ and $\alpha : C \to K(G, 1)$ one can find M_i containing C and an extension $\tilde{\alpha} : M_i \to K(G, 1)$ of α. Since $\tilde{\alpha}$ extends over $S^4 - X_i \supset S^4 - A$, the proof of $\dim_G(S^4 - A) \leq 1$ is completed. □

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