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ALMOST PERIODIC HOMEOMORPHISMS AND p-ADIC TRANSFORMATION GROUPS ON COMPACT 3-MANIFOLDS

JOO S. LEE

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ABSTRACT. In this paper we prove that regularly almost periodic is equivalent to nearly periodic for homeomorphisms on compact metric spaces and give an example to show that the above is false without the compactness assumption. We also prove that the following statement is equivalent to the Hilbert-Smith conjecture on compact 3-manifolds M^3 : If h is almost periodic on M^3, with $h =$ identity on ∂M^3, then $h =$ identity on M^3.

1. INTRODUCTION

Gottschalk [G1] defined a *regularly almost periodic* homeomorphism of a metric space onto itself and proved that a regularly almost periodic homeomorphism on a 2-manifold is periodic. P. A. Smith [S3] defined a *nearly periodic* homeomorphism of a metric space onto itself and conjectured that a nearly periodic homeomorphism on a manifold is periodic. We will prove that regularly almost periodic is equivalent to nearly periodic for homeomorphisms on compact metric spaces (Theorem 3.1) and give an example to show that the compactness is necessary. Later we will give several equivalent conditions to the Hilbert-Smith conjecture (Corollary 3.3 and Theorem 4.2). Hence if we can prove one of those conditions, the Hilbert-Smith conjecture will follow. We also wish to acknowledge receipt of the preprint [M], which asserts that the Hilbert-Smith conjecture is true.

One motivation for this paper is the following question, raised by B. Brechner in [B2]: If h is almost periodic on B^3, with $h =$ identity on ∂B^3, then must h be the identity on B^3? A. Fathi gave me a valuable comment for generalization of Proposition 4.1.

2. DEFINITIONS

A homeomorphism h of a metric space (X, d) onto itself is said to be *almost periodic* [A.P.] on X iff, for every $\epsilon > 0$, there exists a relatively dense sequence $\{n_i\}$ of integers (i.e., the gaps are bounded) such that $d(x, h^{n_i}(x)) < \epsilon$ for all

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$x \in X$ and $i = \pm 1, \pm 2, \pm 3, \ldots$. In particular if, for every $\epsilon > 0$, there exists a positive integer n_ϵ such that $d(x, h^k(x)) < \epsilon$ for all $x \in X$ and for all $k \in n_\epsilon Z$, we say that the homeomorphism h is *regularly almost periodic* [R.A.P.]. Below, we state a well-known characterization (Proposition 2.1) and property (Proposition 2.2) of almost periodic homeomorphisms.

Proposition 2.1 [G2, p. 341]. *Let X be a compact metric space. Then h is an almost periodic homeomorphism on X if and only if the set of powers of h is equicontinuous.*

Proposition 2.2 [G1, p. 55]. *Let h be an almost periodic homeomorphism on a compact metric space (X, d), and let ϵ be any positive number. Then there exists a regularly almost periodic homeomorphism H on X such that $d(h(x), H(x)) < \epsilon$ for each $x \in X$. The homeomorphism H may be chosen as the uniform limit of a sequence of positive powers of h.*

A homeomorphism h of a metric space (X, d) onto itself is said to be *nearly periodic* [N.P.] iff there exists a complete system $\{\Omega_i\}_{i=1}^{\infty}$ of finite covers which are invariant under h. The sequence $\{\Omega_i\}_{i=1}^{\infty}$ is called a *complete system* iff $\{\mathrm{mesh}(\Omega_i)\}$ has limit 0.

3. Main theorem

In this section we will prove our first main theorem (Theorem 3.1) and give an example to show that the compactness hypothesis is necessary. P. A. Smith [S3] showed how to construct a compact zero-dimensional group acting on a compact metric space M, generated by a given N.P. transformation T, of M onto itself. He then stated, without proof, every element of a p-adic transformation group acting on a compact metric space is nearly periodic. In this section, we also provide a proof of this theorem. (See Proposition 3.5.)

Theorem 3.1. *Regularly almost periodic is equivalent to nearly periodic for homeomorphisms on compact metric spaces.*

Proof. Let h be R.A.P. on a compact metric space X. We shall construct a complete system $\{\Omega_i\}_{i=1}^{\infty}$ of finite covers which are invariant under h and such that $\{\mathrm{mesh}(\Omega_i)\}$ has limit 0.

Let $\epsilon_1 > 0$, and choose a finite open cover $B(x_{1,1}, \epsilon_1)$, $B(x_{1,2}, \epsilon_1), \ldots,$ $B(x_{1,k_{\epsilon_1}}, \epsilon_1)$. Since h is R.A.P., there exists a positive integer n_{ϵ_1} such that $d(x, h^n(x)) < \epsilon_1$ for all $n \in n_{\epsilon_1} Z$. We let powers of h act on each $B_{1,k}$, where $B_{1,k} = B(x_{1,k}, \epsilon_1)$. Set

$$B_{1,k}^l = \bigcup_{j=0}^{\infty} h^{jn_{\epsilon_1}+l}(B_{1,k}) \quad \text{for } l = 0, 1, 2, \ldots, n_{\epsilon_1} - 1.$$

Then $\{B_{1,k}^l : l = 0, 1, 2, \ldots, n_{\epsilon_1} - 1\}$ is invariant under h and

$$\{B_{1,k}^l : l = 0, 1, 2, \ldots, n_{\epsilon_1} - 1;\ k = 1, 2, \ldots, k_{\epsilon_1}\}$$

forms a finite invariant open cover of X, denoted by Ω_1.

Now let $\delta_1' > 0$ be the Lebesgue number corresponding to Ω_1. Since the set of powers of h is equicontinuous on X (Proposition 2.1), we can find $\alpha > 0$ such that $h^i(B(x, \alpha)) \subset B(h^i(x), \delta_1')$ for $i \in Z$, $x \in X$. Let δ_1

$= \min\{\delta_1', 1/2^2\}$. Let $\epsilon_2 > 0$ be the number such that $h^i(B(x, \epsilon_2)) \subset B(h^i(x), \delta_1/6)$ for $i \in Z$, $x \in X$; and consider the finite open cover $\{B(x_{2,k}, \epsilon_2) : k = 1, 2, \dots, k_{\epsilon_2}\}$. For convenience, we denote $B(x_{2,k}, \epsilon_2) = B_{2,k}$. Also, from the definition of R.A.P., there exists a positive integer n_{ϵ_2} such that $d(x_{2,k}, h^n(x_{2,k})) < \epsilon_2$ for all $n \in n_{\epsilon_2}Z$. We let powers of h act on $B_{2,k}$ for each k and set

$$B^l_{2,k} = \bigcup_{j=0}^{\infty} h^{jn_{\epsilon_2}+l}(B_{2,k}) \quad \text{for } l = 0, 1, \dots, n_{\epsilon_2} - 1; k = 1, 2, \dots, k_{\epsilon_2}.$$

Then the family of open sets $B^l_{2,k}$, for $l = 0, 1, \dots, n_{\epsilon_2} - 1$, is invariant under h and $\mathrm{diam}(B^l_{2,k}) < \delta_1$ for each l. Therefore, $B^l_{2,k}$ is contained in $B^{l'}_{1,j}$ for some j, l'. Consequently we get the finite open cover

$$\{B^l_{2,k} : l = 0, 1, \dots, n_{\epsilon_2} - 1; k = 1, 2, \dots, k_{\epsilon_2}\} = \Omega_2$$

which is invariant under h, $\mathrm{mesh}(\Omega_2) < \mathrm{mesh}(\Omega_1)$ and $\mathrm{mesh}(\Omega_2) < 1/2^2$.

Inductively we can get a complete system $\{\Omega_i\}$ of finite open covers which are invariant under h and such that $\mathrm{mesh}(\Omega_i) < 1/2^i$.

The converse is clearly true from the definition. □

Theorem 3.1 is not true without the compactness assumption as the following example shows. This example also shows that Proposition 3.5 is false without the compactness assumption.

Example 3.2. Let $\bar{d} = \min\{d(a, b), 1\}$ be a metric on the real line, and define the metric $\rho(\bar{x}, \bar{y})$ on R^∞ to be the l.u.b. $\{\bar{d}(x_i, y_i)\}$ for $\bar{x} = \{x_1, x_2, \dots\}$, $\bar{y} = \{y_1, y_2, \dots\}$. We consider the following tree X which is embedded in R^3 so that the segment $[a, b]$ is located at the z-axis with $d(a, b) = 1/4$ and is contained in the unit disc in the plane. We also define the period two homeomorphism T_2 described in Figure 1 (i.e., T_2 is the π rotation about the z-axis fixing the segment $[a, b]$).

We can define a periodic homeomorphism T_{2^i} on X with period 2^i depending on $T_{2^{i-1}}$.

For example, T_{2^2}, which is the the periodic homeomorphism of X with period four, is the composition of T_2 and the π rotation of subtree below 11, fixing the complement of this subtree and described in Figure 2.

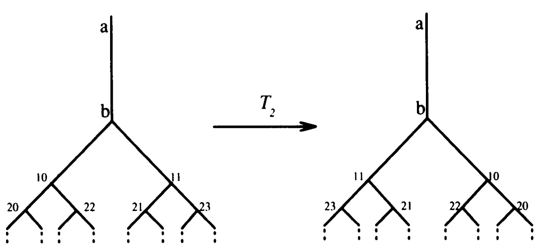

FIGURE 1

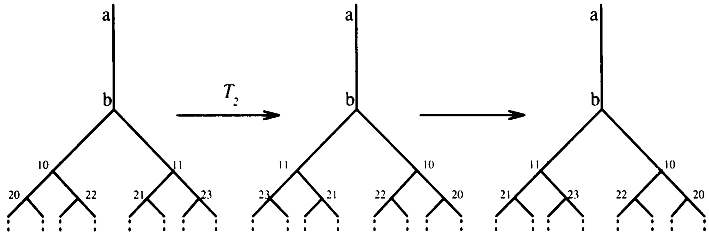

FIGURE 2

Define $T = \lim_{i\to\infty} T_{2^i}$. Then T is R.A.P. and N.P. on X. This shows that we can define N.P. on a metric space without compactness. Now we consider the wedge product

$$X^* = \bigvee_{i=1}^{\infty} X_i \subset \prod (R^3)_i, \quad \text{where } X_i \text{ is the tree described above,}$$

and

$$T^* = \prod_{i=1}^{\infty} T_i : \bigvee_{i=1}^{\infty} X_i \longrightarrow \bigvee_{i=1}^{\infty} X_i, \quad \text{where } T_i = T \text{ on } X_i.$$

Notice that we embed X^* in $\prod_{i=1}^{\infty}(R^3)_i$ so that the wedge point, a, is at the origin. Let $x \in X^*$ with $d(x, 0) > 1/4$. Then, for sufficiently small $\epsilon > 0$, $B_\rho(x, \epsilon) \cap X^* \subset X_j$ for some j. Hence we cannot find a complete system of finite open covers of X^*. But T^* is obviously R.A.P.

Corollary 3.3. *The following statements are equivalent on a compact manifold* M:

(1) *A regularly almost periodic homeomorphism on a compact manifold* M *is periodic.*

(2) *A nearly periodic homeomorphism on a compact manifold* M *is periodic.*

(3) (*Newman's property on regularly almost periodic homeomorphisms.*) *Let* h *be a regularly almost periodic homeomorphism of a compact manifold* M *onto itself. Then there exists* $\epsilon > 0$ *such that if* $i \in Z_+$ *with* $d(x, h^{iZ}(x)) < \epsilon$ *for all* $x \in M$, *then* h^i *acts trivially on* M.

Remark 3.4. Newman's theorem [N] for periodic homeomorphism on manifolds has also been proved by A. Dress [D] and P. A. Smith [S2]. See also [Br, p. 154]. Later H. T. Ku [K] extended Newman's theorem to actions of p-adic solenoids.

Proposition 3.5. *Let* X *be a compact metric space and* G *be a p-adic transformation group acting on* X. *Then every element of* G *is nearly periodic.*

Proof. We use the technique of Yang [Y, p. 211]. Let X be a compact metric space and G be a p-adic group acting as a topological transformation group on X. Let $G = G_0 \supset G_1 \supset \cdots$ be a sequence of open subgroups of G such that, whenever $j > i$, G_i/G_j is a cyclic group of order p^{j-i} and

$$\lim_{i\to\infty} \{\text{diam}(G_i)\} = 0.$$

Let

$$h_{i,j}\colon G/G_j \longrightarrow G/G_i\,, \qquad h_i\colon G \longrightarrow G/G_i$$

be homomorphisms induced by the identity homomorphism of G; i.e., $gG_j \to gG_i$, $g \to gG_i$ by $h_{i,j}$, h_i respectively. Then $G \simeq \lim_{\leftarrow}\{G/G_j\}$ with bonding map $h_{i,j}$. Similarly, we let

$$\pi_{i,j}\colon X/G_j \longrightarrow X/G_i\,, \qquad \pi_i\colon X \longrightarrow X/G_i$$

be maps induced by the identity homeomorphism of X. Then $\{X/G_i : \pi_{i,j}\}$ is an inverse system and $\{\pi_i\}$ determines a homeomorphism of X onto $\lim_{\leftarrow}\{X/G_j\}$ by $x \to (xG_1, xG_2, \dots)$. Note that X/G_i denotes the orbit space of X determined by G_i.

Let T be an element of G, and for every nonnegative integer i, let T_i be the coset TG_i in G/G_i. Since TG_i acts on X/G_i, T_i is a periodic map on X/G_i with period p^k, where $k \le i$. Let T_j be the periodic map for $j > i$. Then the period of T_j is no less than the period of T_i since $G_i \supset G_j$. We assume that the period of T_1 is p. Let

$$\Psi_1 = \{U_{1,1}, U_{1,2}, \dots, U_{1,k_1}\}$$

be a finite open cover of X such that Ψ_1 is π_1-saturated for some finite open cover of X/G_1. We let powers of T_1 act on Ψ_1 and find the finite open cover

$$\Omega_1 = \{\Psi, T_1(\Psi), \dots, T_1^{p-1}(\Psi)\}$$

which is invariant under T.

Now let $\delta_1{}'$ be the Lebesgue number of Ω_1, $\delta_1 = \min\{\delta_1{}', 1/2^2\}$, and let $\epsilon_2 > 0$ be a number such that $T^i(B(x, \epsilon_2)) \subset B(T^i(x), \delta_1/6)$ for all i, for all $x \in X$. Let G_k be a sufficiently small subgroup such that

$$\Psi_k = \{U_{k,1}, U_{k,2}, \dots, U_{k,l_k}\}$$

is a finite open cover of X with $\operatorname{diam}(U_{k,i}) < \epsilon_2$ and π_k-saturated for some finite open cover of X/G_k. Then

$$\Omega_k = \{\Psi_k, T_k(\Psi_k), \dots, T_k^{q-1}(\Psi_k)\}$$

where q is the period of T_k, is an invariant finite open cover of X under T and $\operatorname{mesh}(\Omega_k) < \delta_1$. Thus we get the finite open cover Ω_k which is invariant under T and $\operatorname{mesh}(\Omega_k) < 1/2^2$.

Inductively we get a complete system $\{\Omega_i\}$ of finite open covers which are invariant under T, and $\operatorname{mesh}(\Omega_i)$ has limit 0. $\square$

4. The Hilbert-Smith conjecture on compact 3-manifolds

M. H. A. Newman [N], P. A. Smith [S1], and A. Dress [D] proved that if G is a compact Lie group acting effectively on a manifold M, then the fixed-point set is nowhere dense. From this result we have the following proposition:

Proposition 3.1. *Let M be a compact, connected manifold with nonempty boundary, and let h be a periodic homeomorphism of M onto itself such that h is the identity on ∂M. Then h is the identity on M.*

Proof. We attach two copies of M on their boundaries with the identity map and let H be the map of $M \cup_{\mathrm{id}\,|\partial M} M$ onto itself such that $H = h$ on one copy of M and $H =$ identity on the other copy of M. Then H is periodic on this double of M and the fixed-point set contains an open subset. Therefore $h =$ identity by Newman's theorem. □

In general, A.P. and R.A.P. are not equivalent on a metric space . For example, an irrational rotation on S^1 or D^2 is A.P. but is not R.A.P. [F, vK]. But, from Proposition 3.5, Proposition 4.1, and Corollary 3.3, we have the following theorem:

Theorem 4.2. *Let M^3 be a compact, connected 3-manifold with nonempty boundary, and let h be a homeomorphism of M^3 onto itself. Then each of the following statements is equivalent to the Hilbert-Smith conjecture on M^3:*

(1) *If h is almost periodic on M^3, with $h =$ identity on ∂M^3, then $h =$ identity on M^3.*

(2) *If h is regularly almost periodic on M^3, with $h =$ identity on ∂M^3, then $h =$ identity on M^3.*

(3) *If h is regularly almost periodic on M^3, then h is periodic on M^3.*

(4) *If h is nearly periodic on M^3, then h is periodic on M^3.*

(5) *(Newman's property on regularly almost periodic homeomorphisms.) Let h be a regularly almost periodic homeomorphism of M^3 onto itself. Then there exists $\epsilon > 0$ such that if $i \in Z_+$ with $d(x, h^{iZ}(x)) < \epsilon$ for all $x \in M^3$, then h^i acts trivially on M^3.*

Proof. (1) implies (2): Clear.

(2) implies (3): Let h be R.A.P. on M^3. Then h is periodic on ∂M^3 [G1]. Let n be the period of h on ∂M^3. Then h^n is R.A.P. and identity on ∂M^3. Therefore, h^n is the identity on M^3, by hypothesis, and hence h is periodic on M^3.

(3) implies (1): Let h be A.P. with $h =$ identity on ∂M^3 and let $\epsilon > 0$. By Proposition 2.2 there exists a R.A.P. homeomorphism H on M^3 such that $d(h(x), H(x)) < \epsilon$ for each $x \in M^3$. H is periodic by assumption. Since H is a uniform limit of a sequence of positive powers of h and $h =$ identity on ∂M^3, $H =$ identity on ∂M^3. Therefore, $H =$ identity on M^3 by Proposition 4.1. Since ϵ was arbitrary, $h =$ identity on M^3.

By Corollary 3.3, (3), (4), and (5) are equivalent. □

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Department of Mathematics, University of Florida, Gainesville, Florida 32611

Current address: Department of Mathematics, Dongguk University, Chung-Ku Seoul, Korea 100-273

E-mail address: `jsl@math.ufl.edu`