PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 2, June 1994

A CHARACTERIZATION OF M_p-GROUPS

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(Communicated by Ronald M. Solomon)

ABSTRACT. We characterize finite groups, all irreducible characters of which over an algebraically closed field of characteristic p are monomial.

1. INTRODUCTION

Let F be an algebraically closed field of characteristic $p > 0$, and let G be a finite group. We call G an M_p-group if every irreducible FG-module is monomial. This is defined by Okuyama [2], and it is a p-modular analogue of the definition of M-groups.

In [3] Parks provided a purely group theoretic characterization of M-groups. In this paper, we will provide a characterization of M_p-groups, which is based on the result by Parks and unfortunately not purely group theoretic.

2. RESULT

Parks has introduced the concept of good pairs and a relation on the set of them. Using them, we define p'-good pairs and a relation. Let H and K be subgroups of G such that $K \triangleleft H$ and H/K is cyclic. Then we call (H, K) a pair, and if H/K is of p'-order then we call (H, K) a p'-pair. For $g \in G$ and $H \leq G$, we define $F_H(g)$ to be the set of commutators $[g, H \cap H^{g^{-1}}]$. If (H, K) is a pair and $F_H(g) \nleq K$ for all $g \in G - H$, we call it a good pair in G. We call (H, K) a p'-good pair if (H, K) is a good pair and a p'-pair.

For good pairs (H, K) and (L, N), we will say they are related in G if there exists $g \in G$ such that $H^g \cap N = L \cap K^g$. In general, being related is an equivalence relation on the set of good pairs. This is proved in [3]. We restrict this to the set of p'-good pairs in G. Then being related is also an equivalence relation on the set of p'-good pairs. Let $S_{G,p}$ be the equivalence relation on p'-good pairs in G generated by the relation of being related, and let $m_{G,p}$ be the number of distinct classes of $S_{G,p}$.

We say $x \sim y$ for $x, y \in G$ if two cyclic subgroups $\langle x \rangle$ and $\langle y \rangle$ are conjugate in G. We restrict the relation $\sim$ to the set of p'-elements in G. Let $n_{G,p}$ be the number of $\sim$ equivalence classes of p'-elements in G.

Now we can state our main theorem, and this is a p-modular analogue of the result by Parks.

Received by the editors June 8, 1992 and, in revised form, September 15, 1992.
1991 *Mathematics Subject Classification*. Primary 20C20.

Theorem. *Let G be a finite group. Then G is an M_p-group if and only if the following three conditions hold*:

(a) $m_{G,p} = n_{G,p}$.
(b) *Every irreducible FG-module is a trivial source module.*
(c) *G is p-solvable.*

Remark. (i) M_p-groups are solvable [2, Corollary 3.8].

(ii) If G is a p'-group then the conditions (b) and (c) are clearly satisfied, and so this is the same as the result by Parks.

(iii) The condition (b) is not group theoretic, but we need it. For example, $\mathrm{GL}(2, 3)$ satisfies the conditions (a) and (c) for $p = 3$, but it is not an M_3-group.

3. Proof

For a pair (H, K), there exists a linear character λ of H such that $\ker\lambda = K$. We will say λ proceeds from (H, K). Similarly, for a p'-pair (H, K), there exists a one-dimensional FH-module W with $\ker W = K$. We will say W proceeds from (H, K). Parks has shown that a pair (H, K) is a good pair in G if and only if λ^G is irreducible for λ proceeding from (H, K) [3, Proposition 1.1]. For modules, this is not true. We can only say that $\dim_F \mathrm{End}_{FG}(W^G) = 1$. In fact, for $G = \mathrm{GL}(2, 3)$ and $p = 3$, there exist a p'-good pair and W proceeding from it such that W^G is reducible. But, if we assume the condition (b) in Theorem, W^G is always irreducible (Lemma 1).

Let (K, R, F) be a splitting p-modular system for all finite groups, and let (π) be the unique maximal ideal of R. Then, for a trivial source FG-module M, there exists a trivial source RG-module $\widetilde{M}$ such that $\widetilde{M}/\pi\widetilde{M} \cong M$, and this is uniquely determined. We write χ_M for the character of $\widetilde{M} \otimes_R K$. In this sense, χ_M is uniquely determined by M. For trivial source FG-modules M and N, the following is a well-known fact:

$$\dim_F \mathrm{Hom}_{FG}(M, N) = (\chi_M, \chi_N)_G$$

where $(\chi_M, \chi_N)_G = \frac{1}{|G|} \sum_{g\in G} \chi_M(g)\overline{\chi_N}(g)$.

If M proceeds from a p'-pair (H, K), then clearly M is a trivial source FH-module and χ_M proceeds from (H, K). Also it is easy to see that $(\chi_M)^G = \chi_{M^G}$.

Lemma 1. *Let (H, K) be a p'-pair in G, and let W proceed from (H, K). Assume that every irreducible FG-module is a trivial source module. Then (H, K) is a p'-good pair if and only if W^G is irreducible.*

Proof. It is clear that if W^G is irreducible then (H, K) is a p'-good pair, since χ_{W^G} is irreducible. Assume that (H, K) is a p'-good pair and W^G is reducible. Then, for some irreducible FG-module S,

$$\dim_F \mathrm{Hom}_{FG}(W^G, S) \neq 0.$$

Note that W and S are trivial source modules and $(\chi_W)^G = \chi_{W^G}$. We have

$$((\chi_W)^G, \chi_S)_G = \dim_F \mathrm{Hom}_{FG}(W^G, S) \neq 0.$$

The character χ_W proceeds from (H, K) and (H, K) is a good pair, so $(\chi_W)^G$ is irreducible. Thus $(\chi_W)^G = \chi_S$ and so $W^G \cong S$. This contradicts to the assumption. □

Lemma 2. *Let (H, K) and (L, N) be p'-good pairs. Assume that every irreducible FG-module is a trivial source module. Then they are related if and only if there are modules W and U proceeding from (H, K) and (L, N), respectively, such that $W^G \cong U^G$.*

Proof. This is immediate from Lemma 1 and the proof of [3, Proposition 1.2]. □

Let m be the p'-part of the order of G, and let φ_M and φ_N be the Brauer characters of irreducible FG-modules M and N, respectively (these depend on the choice of primitive mth roots of unities in F and in the complex number field $\mathbb{C}$). We will say that M and N are Galois conjugate if there is $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that $(\varphi_M)^\sigma = \varphi_N$ (this is not standard). Let ζ be a primitive mth root of unity in $\mathbb{C}$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, there is an integer n such that $\zeta^\sigma = \zeta^n$, and n is uniquely determined in mod m. For a p'-element $g \in G$, we define $g^\sigma = g^n$. Then it is easy to see that $\varphi^\sigma(g) = \varphi(g^\sigma)$ for any Brauer character φ. Assume that G is p-solvable. Then φ^σ is also an irreducible Brauer character of G [1, Theorem 2F]. Thus we can apply the Brauer's Permutation Lemma to the Brauer character table for G. So the number of Galois conjugate classes of irreducible FG-modules is equal to $n_{G,p}$ defined above.

Proof of Theorem. If G is an M_p-group then clearly (b) and (c) hold. Assume that G is p-solvable and every irreducible FG-module is a trivial source module. For a p'-good pair (H, K), an FH-module M proceeding from (H, K) is uniquely determined up to Galois conjugate, and so is M^G. So an equivalence class of p'-good pairs determines a Galois conjugate class of irreducible monomial FG-modules. Also a Galois conjugate class of irreducible monomial FG-modules determines an equivalence class of p'-good pairs. So the correspondence is one-to-one, and the number of their classes is $m_{G,p}$. On the other hand, the number of Galois conjugate classes of irreducible FG-modules is equal to $n_{G,p}$. Now the theorem holds obviously. □

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