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# DERIVATIONS IN PRIME NEAR-RINGS

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ABSTRACT. Let $N$ be a prime near-ring with center $Z$. The purpose of this paper is to study derivations on $N$. We show two main results: (1) Let $N$ be 2-torsion-free, and let $D_1$ and $D_2$ be derivations on $N$ such that $D_1D_2$ is also a derivation. Then either $D_1$ or $D_2$ is zero if and only if $[D_1(x), D_2(y)] = 0$ for all $x, y \in N$. (2) Let $n$ be an integer $\geq 1$, $N$ be $n!$-torsion-free, and $D$ a derivation with $D^n(N) = \{0\}$. Then $D(Z) = \{0\}$.

Throughout this paper $N$ always stands for a zero-symmetric left near-ring. An additive endomorphism $D$ of $N$ is called a derivation on $N$ if $D(xy) = xD(y) + D(x)y$ for all $x, y \in N$. According to [1], a near-ring $N$ is said to be prime if $xNy = \{0\}$ for $x, y \in N$ implies $x = 0$ or $y = 0$.

As the addition of a near-ring is not necessarily commutative, the following Proposition 1 has its own significance.

**Proposition 1.** *Let $D$ be an arbitrary additive endomorphism of $N$. Then $D(xy) = xD(y)+D(x)y$ for all $x, y \in N$ if and only if $D(xy) = D(x)y+xD(y)$ for all $x, y \in N$. Therefore $D$ is a derivation if and only if $D(xy) = D(x)y + xD(y)$.*

*Proof.* Suppose $D(xy) = xD(y) + D(x)y$ for all $x, y \in N$. Since $x(y + y) = xy + xy$ and

$$D(x(y + y)) = xD(y + y) + D(x)(y + y) = xD(y) + xD(y) + D(x)y + D(x)y$$

and

$$D(xy + xy) = D(xy) + D(xy) = xD(y) + D(x)y + xD(y) + D(x)y,$$

we get $xD(y) + D(x)y = D(x)y + xD(y)$, so $D(xy) = D(x)y + xD(y)$.

The converse is proved in a similar way.

**Lemma 1.** *Let $D$ be an arbitrary derivation on $N$. Then $N$ satisfies the following partial distributive laws (for $x, y, z, \in N$):*

(i) $(xD(y) + D(x)y)z = xD(y)z + D(x)yz$;
(ii) $(D(x)y + xD(y))z = D(x)yz + xD(y)z$.

*Proof.* (i) was proved in [1], so we only need to prove (ii). From the associative law and Proposition 1 we have

$$D((xy)z) = D(xy)z + xyD(z) = (D(x)y + xD(y))z + xyD(z)$$

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and

$$D(x(yz)) = D(x)yz + xD(yz) = D(x)yz + x(D(y)z + yD(z))$$
$$= D(x)yz + xD(y)z + xyD(z).$$

Comparing the two expressions we obtain

$$(D(x)y + xD(y))z = D(x)yz + xD(y)z.$$

Now we prove our first main result, which extends a famous theorem on rings of Posner [2] to near-rings.

**Theorem 1.** *Let $N$ be a 2-torsion-free prime near-ring, and let $D_1$ and $D_2$ be derivations on $N$ such that $D_1D_2$ is also a derivation. Then the following two conditions are equivalent:*

(i) *either $D_1 = 0$ or $D_2 = 0$;*
(ii) $[D_1(x), D_2(y)] = 0$ *for all $x, y \in N$.*

*Proof.* (i) $\Rightarrow$ (ii) is obvious. We only prove (ii) $\Rightarrow$ (i). Noting that $D_1D_2$ is a derivation, we have

$$D_1D_2(xy) = xD_1D_2(y) + D_1D_2(x)y.$$

On the other hand, $D_1$ and $D_2$ are both derivations, so

$$D_1D_2(xy) = D_1(D_2(xy)) = D_1(xD_2(y) + D_2(x)y)$$
$$= D_1(xD_2(y)) + D_1(D_2(x)y)$$
$$= xD_1D_2(y) + D_1(x)D_2(y) + D_2(x)D_1(y) + D_1D_2(x)y.$$

The above two expressions for $D_1D_2(xy)$ yield

(1) $$D_1(x)D_2(y) + D_2(x)D_1(y) = 0 \quad \text{for all } x, y \in N.$$

Replacing $x$ by $xD_2(z)$ in (1), by using Proposition 1 and Lemma 1 we have

$$0 = D_1(xD_2(z))D_2(y) + D_2(xD_2(z))D_1(y)$$
$$= (D_1(x)D_2(z) + xD_1D_2(z))D_2(y) + (xD_2^2(z) + D_2(x)D_2(z))D_1(y)$$
$$= D_1(x)D_2(z)D_2(y) + xD_1D_2(z)D_2(y) + xD_2^2(z)D_1(y) + D_2(x)D_2(z)D_1(y)$$
$$= D_1(x)D_2(z)D_2(y) + x(D_1D_2(z)D_2(y) + D_2^2(z)D_1(y)) + D_2(x)D_2(z)D_1(y).$$

In this equality

$$x(D_1D_2(z)D_2(y) + D_2^2(z)D_1(y)) = 0$$

because the second factor $D_1D_2(z)D_2(y) + D_2^2(z)D_1(y) = 0$ by equality (1) with $x$ replaced by $D_2(z)$. Thus we get

(2) $$D_1(x)D_2(z)D_2(y) + D_2(x)D_2(z)D_1(y) = 0 \quad \text{for all } x, y, z, \in N.$$

Replacing $x$ and $y$ by $z$ in (1), respectively, we obtain

$$D_2(z)D_1(y) = -D_1(z)D_2(y)$$

and

$$D_1(x)D_2(z) = -D_2(x)D_1(z).$$

Since $N$ is a zero-symmetric left near-ring, (2) becomes

$$\begin{aligned}0 &= (-D_2(x)D_1(z))D_2(y) + D_2(x)(-D_1(z)D_2(y)) \\ &= D_2(x)(-D_1(z))D_2(y) + D_2(x)(-D_1(z)D_2(y)) \\ &= D_2(x)[(-D_1(z))D_2(y) - D_1(z)D_2(y)]\end{aligned}$$

for all $x, y, z \in N$. If $D_2 \neq 0$, by [1, Lemma 3] we have

$$(-D_1(z))D_2(y) - D_1(z)D_2(y) = 0,$$

that is,

(3) $$D_1(z)D_2(y) = (-D_1(z))D_2(y) \quad \text{for all } y, z \in N.$$

However, by condition (ii) we have

$$\begin{aligned}(-D_1(z))D_2(y) &= D_1(-z)D_2(y) = D_2(y)D_1(-z) \\ &= D_2(y)(-D_1(z)) = -D_2(y)D_1(z) = -D_1(z)D_2(y),\end{aligned}$$

that is,

(4) $$(-D_1(z))D_2(y) = -D_1(z)D_2(y) \quad \text{for all } y, z \in N.$$

From (3) and (4) we obtain

$$2D_1(z)D_2(y) = 0,$$

or $D_1(z)D_2(y) = 0$ since $N$ is 2-torsion-free. Hence $D_1(z)D_2(N) = \{0\}$, but $D_2 \neq 0$ so $D_1(z) = 0$ for all $z \in N$. Thus $D_1 = 0$.

As a consequence of Theorem 1, we obtain

**Corollary 1** [1]. *Let $N$ be a 2-torsion-free prime near-ring, and let $D$ be a derivation on $N$ such that $D^2 = 0$. Then $D = 0$.*

*Proof.* It is clear that $D^2 = 0$ is a derivation on $N$, and we have

$$\begin{aligned}0 &= D^2(xy) = D(xD(y) + D(x)y) = D(xD(y)) + D(D(x)y) \\ &= xD^2(y) + D(x)D(y) + D(x)D(y) + D^2(x)y = 2D(x)D(y),\end{aligned}$$

so $D(x)D(y) = 0$. Therefore $[D(x), D(y)] = 0$ for all $x, y \in N$. From Theorem 1, $D = 0$.

Using equality (1) in the proof of Theorem 1 we can prove the following interesting result.

**Proposition 2.** *Let $N$ be a near-ring and $D_1$ and $D_2$ be derivations on $N$ such that $D_1D_2$ is a derivation. Then $D_2D_1$ is also a derivation.*

*Proof.* Obviously $D_2D_1$ is an additive endomorphism of $N$. By equality (1) and Proposition 1 we have

$$\begin{aligned}D_2D_1(xy) &= D_2(D_1(x)y + xD_1(y)) = D_2(D_1(x)y) + D_2(xD_1(y)) \\ &= D_2D_1(x)y + (D_1(x)D_2(y) + D_2(x)D_1(y)) + xD_2D_1(y) \\ &= D_2D_1(x)y + xD_2D_1(y)\end{aligned}$$

for all $x, y \in N$. Thus $D_2D_1$ is a derivation by Proposition 1.

Corollary 1 leads us naturally to ask a question: let integer $n \geq 2$, and let $N$ be an $n!$-torsion-free prime near-ring. If $D$ is a derivation on $N$ such that $D^n(N) = \{0\}$, can we conclude $D(N) = \{0\}$?

The answer is negative even for rings. A simple counterexample due to Chung, Kobayashi, and Luh [3] is as follows: Let $R$ be the ring of $2 \times 2$ matrices over $GF(P)$, where $P$ is a prime integer greater than 3 and $D$ be the inner derivation induced by $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Then $N$ is 3!-torsion-free and $D^3(R) = \{0\}$, but $D(R) \neq \{0\}$.

Nevertheless we will show that in the case of near-ring $D(Z) = \{0\}$ where $Z$ is the center of $N$.

In order to discuss the question we need to extend Leibniz' rule for derivations of rings to near-rings.

**Proposition 3.** *Leibniz' rule holds in near-rings, namely, for any integer $n \geq 2$ and any $x, y \in N$, it holds that*

$$D^n(xy) = D^n(x)y + \binom{n}{1} D^{n-1}(x)D(y) + \cdots + \binom{n}{i} D^{n-i}(x)D^i(y) + \cdots + \binom{n}{n-1} D(x)D^{n-1}(y) + xD^n(y).$$

*Proof.* Using Proposition 1 and elementary facts about centralizers of elements in group, one can easily prove

$$D(x)y + nxD(y) = nxD(y) + D(x)y.$$

Further, we can prove

$$nD(x)y + nxD(y) = n(D(x)y + xD(y)) \quad \text{for all } x, y \in N. \tag{5}$$

Next we prove Leibniz' rule by induction on $n$. When $n = 2$ we have

$$\begin{aligned} D^2(xy) &= D(D(x)y + xD(y)) \\ &= D(D(x)y) + D(xD(y)) \\ &= D^2(x)y + D(x)D(y) + D(x)D(y) + xD^2(y) \\ &= D^2(x)y + 2D(x)D(y) + xD^2(y). \end{aligned}$$

Assume Leibniz' rule holds for $n - 1$. That is, if $N$ is $(n-1)!$-torsion-free, then

$$D^{n-1}(xy) = D^{n-1}(x)y + \cdots + \binom{n-1}{i-1} D^{n-i}(x)D^{i-1}(y) + \binom{n-1}{i} D^{n-i-1}(x)D^i(y) + \cdots + xD^{n-1}(y).$$

Since $n!$-torsion-freeness implies $(n-1)!$-torsion-freeness, by (5) we have

$$\begin{aligned}
D^n(xy) &= D(D^{n-1}(xy)) \\
&= D\Big(D^{n-1}(x)y + \cdots + \binom{n-1}{i-1}D^{n-i}(x)D^{i-1}(y) \\
&\qquad\qquad + \binom{n-1}{i}D^{n-i-1}(x)D^{i}(y) + \cdots + xD^{n-1}(y)\Big) \\
&= D(D^{n-1}(x)y) + \cdots + \binom{n-1}{i-1}D(D^{n-i}(x)D^{i-1}(y)) \\
&\quad + \binom{n-1}{i}D(D^{n-i-1}(x)D^{i}(y)) + \cdots + D(xD^{n-1}(y)) \\
&= D^n(x)y + D^{n-1}(x)D(y) \\
&\quad + \cdots + \binom{n-1}{i-1}(D^{n-i+1}(x)D^{i-1}(y) + D^{n-i}(x)D^{i}(y)) \\
&\quad + \binom{n-1}{i}(D^{n-i}(x)D^{i}(y) + D^{n-i-1}(x)D^{i+1}(y)) \\
&\quad + \cdots + D(x)D^{n-1}(y) + xD^n(y) \\
&= D^n(x)y + \cdots + \binom{n-1}{i-1}D^{n-i+1}(x)D^{i-1}(y) + \binom{n-1}{i-1}D^{n-i}(x)D^{i}(y) \\
&\quad + \binom{n-1}{i}D^{n-i}(x)D^{i}(y) + \binom{n-1}{i}D^{n-i-1}(x)D^{i+1}(y) + \cdots + xD^n(y) \\
&= D^n(x)y + \cdots + \binom{n-1}{i-1}D^{n-i}(x)D^{i}(y) + \binom{n-1}{i}D^{n-i}(x)D^{i}(y) \\
&\quad + \cdots + xD^n(y) \\
&= D^n(x)y + \cdots + \left[\binom{n-1}{i-1} + \binom{n-1}{i}\right]D^{n-i}(x)D^{i}(y) \\
&\quad + \cdots + xD^n(y) \\
&= D^n(x)y + \cdots + \binom{n}{i}D^{n-i}(x)D^{i}(y) + \cdots xD^n(y).
\end{aligned}$$

The proof is completed.

**Lemma 2.** *Let $N$ be a near-ring with center $Z$, and let $D$ be a derivation on $N$. Then $D(Z) \subseteq Z$.*

*Proof.* From Proposition 1, for any $z \in Z$ and any $x \in N$ we have

$$xD(z) + zD(x) = xD(z) + D(x)z = D(xz) = D(zx) = D(z)x + zD(x).$$

It follows that $xD(z) = D(z)x$, that is, $D(z) \in Z$.

**Lemma 3.** *Let $n \geq 2$, and let $N$ be an $n!$-torsion-free near-ring and $D$ be a derivation with $D^n(N) = \{0\}$. Then for each $y \in N$, either $D(y) = 0$ or there exists $0 < k < n$ such that $D^k(y)$ is a nonzero divisor of zero.*

*Proof.* Since $n!$-torsion-freeness implies $(n-1)!$-torsion-freeness, we may assume that $D^{n-1}(N) \neq \{0\}$, in which case we choose $x_0$ such that $D^{n-1}(x_0) \neq 0$. Assume $D(y) \neq 0$. Then there exists $k$ with $0 < k < n$ for which $D^k(y) \neq 0$ and $D^{k+1}(y) = 0$.

Using Leibniz' rule we obtain

$$\begin{aligned}0 = D^n(x_0D^{k-1}(y)) &= D^n(x_0)D^{k-1}(y) + \binom{n}{1}D^{n-1}(x_0)D^k(y)\\ &\quad + \binom{n}{2}D^{n-2}(x_0)D^{k+1}(y) + \cdots\\ &= \binom{n}{1}D^{n-1}(x_0)D^k(y) = nD^{n-1}(x_0)D^k(y).\end{aligned}$$

We get $D^{n-1}(x_0)D^k(y) = 0$ since $N$ is $n!$-torsion-free. So $D^k(y)$ is a nonzero divisor of zero.

Now we can prove our second main theorem.

**Theorem 2.** *Let $n$ be an integer $\geq 1$ and $N$ be a prime near-ring with center $Z$, and let $N$ be $n!$-torsion-free and $D$ a derivation with $D^n(N) = \{0\}$. Then $D(Z) = \{0\}$.*

*Proof.* If $n = 1$, there is nothing to prove. If $n \geq 2$, suppose $D(Z) \neq \{0\}$. We choose $z \in Z$ such that $D(z) \neq 0$. By Lemmas 2 and 3, there exists a positive integer $k$ such that $D^k(z)$ is a nonzero divisor of zero contained in $Z$. On the other hand, $D(z)$ could not be a divisor by [1, Lemma 3]. The contradiction proves that $D(Z) = \{0\}$.

Finally we drop the condition that $N$ is prime to obtain the following

**Theorem 3.** *Let $n$ be a positive integer and $N$ be an $n!$-torsion-free near-ring with no divisor of zero, then $N$ admits no nonzero derivation $D$ with $D^n = 0$.*

The proof is immediately obtained by Lemma 3.

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## References

1. H. E. Bell and G. Mason, *On derivations in near-ring*, Near-Rings and Near-Fields (G. Betsch, ed.), North-Holland, Amsterdam, 1987, pp. 31–35.
2. E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.
3. L. O. Chung, Y. Kobayashi, and J. Luh, *Remark on nilpotency of derivations*, Proc. Japan Acad. Math. Sci. Ser. A **60** (1984), 329–330.

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