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$n\times$ OVERSAMPLING PRESERVES ANY TIGHT AFFINE FRAME FOR ODD n

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ABSTRACT. If ψ generates an affine frame $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$, of $L^2(\mathbb{R})$, we prove that $\{n^{-1/2}\psi_{j,k/n}\}$ is also an affine frame of $L^2(\mathbb{R})$ with the same frame bounds for any positive odd integer n. This establishes the result stated as the title of this paper. A counterexample of this statement for $n = 2$ is also given.

1. INTRODUCTION AND RESULTS

Let $L^2 = L^2(\mathbb{R})$ denote, as usual, the space of all complex-valued square-integrable functions on the real line with inner product $\langle\ ,\ \rangle$ and norm $\|\ \|$. For any $f \in L^2$, we will use the notation

$$f_{j,\alpha}(x) = 2^{j/2}f(2^j x - \alpha), \qquad j \in \mathbb{Z},\ \alpha \in \mathbb{R}. \tag{1.1}$$

A function $\psi \in L^2$ is said to generate an *affine frame*

$$\{\psi_{j,k}: j, k \in \mathbb{Z}\} \tag{1.2}$$

of L^2, with *frame bounds* A and B, where $0 < A \leq B < \infty$, if it satisfies

$$A\|f\|^2 \leq \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 \leq B\|f\|^2, \qquad f \in L^2. \tag{1.3}$$

The frame (1.2) of L^2 is called a *tight frame*, if (1.3) holds with $A = B$. The importance of a tight frame is that any $f \in L^2$ can be recovered from its *integral wavelet transform* (IWT)

$$\langle f, \psi_{j,k}\rangle = 2^{j/2}\int_{-\infty}^{\infty} f(x)\overline{\psi\left(2^j\left(x - \frac{k}{2^j}\right)\right)}\,dx \tag{1.4}$$

relative to ψ at the time-scale locations $(2^{-j}, k/2^j)$, $j, k \in \mathbb{Z}$, via the formula

$$f(x) = \frac{1}{A}\sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle\psi_{j,k}(x), \tag{1.5}$$

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where $A = B$. It should be noted that a frame, such as (1.2)–(1.3), tight or not, may be redundant in the sense that (1.2) does not have to be l^2-linearly independent. However, any Riesz (or unconditional) basis is also a frame.

In the above discussion, we only consider, without loss of generality, the sampling period $b = 1$ and scaling parameter $a = 2$ as in (1.2). Details and generality are discussed in the wavelet literature [1, 2, 3, 5, 6], and a general study of frames can be found in the monograph [7] as well as the fundamental paper [4] of Duffin and Schaeffer, where the notion of frames was first introduced.

The objective of this paper is to establish the following.

Theorem 1. *Let $\psi \in L^2$ generate a frame $\{\psi_{j,k}\}$ of L^2 with frame bounds A and B as given by* (1.3). *Then for any positive odd integer n, the family*

$$\{n^{-1/2}\psi_{j,k/n}\colon j, k \in \mathbb{Z}\} \tag{1.6}$$

remains a frame of L^2 with the same bounds: that is,

$$nA\|f\|^2 \le \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k/n}\rangle|^2 \le nB\|f\|^2, \qquad f \in L^2. \tag{1.7}$$

In particular, if $\{\psi_{j,k}\}$ is a tight frame (*with $A = B$*) *and $n > 0$ is odd, then the family in* (1.6) *satisfies*

$$\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k/n}\rangle|^2 = nA\|f\|^2, \qquad f \in L^2. \tag{1.8}$$

On the other hand, (1.8) *does not necessarily hold for even $n > 0$.*

2. Preliminary results

A sequence of three lemmas will be needed for the proof of Theorem 1.

Let n be any positive odd integer and set

$$\lambda_1(p) = 2p - \frac{n}{2}(1 + \operatorname{sgn}(2p - n)). \tag{2.1}$$

Then λ_1 is a permutation of the set $\{0, \dots, n-1\}$. This permutation gives rise to a rearrangement operator τ defined on $\mathbb{R}^n$ by

$$\tau(\mathbf{a}) := (a_{\lambda_1(0)}, \dots, a_{\lambda_1(n-1)}), \qquad \mathbf{a} = (a_0, \dots, a_{n-1}) \in \mathbb{R}^n, \tag{2.2}$$

where $n > 0$ is odd. As usual, we set

$$\tau^0 = I \quad \text{and} \quad \tau^j = \tau(\tau^{j-1}), \qquad j = 1, 2, \dots, \tag{2.3}$$

with I denoting the identity operator. Hence, for each $j = 0, 1, \dots$, we may write

$$\tau^j(\mathbf{a}) = (a_{\lambda_j(0)}, \dots, a_{\lambda_j(n-1)}), \qquad \mathbf{a} \in \mathbb{R}^n, \tag{2.4}$$

where λ_j is a permutation of $\{0, \dots, n-1\}$ induced by λ_1. We have the following result.

Lemma 1. *Let* n *be any positive odd integer and* λ_j *be defined by* (2.1)–(2.4). *Then for any* $j \in \mathbb{Z}_+$,

$$\lambda_j(p) \equiv 2^j p \pmod{n}, \qquad p = 0, \dots, n-1. \tag{2.5}$$

Proof. We will establish (2.5) by induction on j. Since λ_0 is the identity, (2.5) certainly holds for $j = 0$. For $j \geq 1$, we first consider $0 \leq p \leq (n-1)/2$. In this case, it follows from (2.1) that $\lambda_{j+1}(p) = \lambda_j(2p)$. Hence, by the induction hypothesis, we have

$$\lambda_{j+1}(p) = \lambda_j(2p) \equiv 2^j(2p) \ (\mathrm{mod}\ n) = 2^{j+1}p \ (\mathrm{mod}\ n).$$

Similarly, for $(n-1)/2 < p \leq n-1$, it follows from (2.1) and the induction hypothesis that

$$\lambda_{j+1}(p) = \lambda_j(2p-n) \equiv 2^j(2p-n) \ (\mathrm{mod}\ n) \equiv 2^{j+1}p \ (\mathrm{mod}\ n). \quad \square$$

Since λ_1 is a one-one map of $\{0, \dots, n-1\}$ onto itself, the rearrangement operator τ as defined by (2.2) has an inverse τ^{-1}. Hence, the definition of τ^j in (2.3) can be extended to all $j \in \mathbb{Z}$. Set

$$\mathbf{a}_0 := (0, 1, \dots, n-1)$$

and define $\{\varepsilon_{j,p}\}$, $p = 0, \dots, n-1$, and $j \in \mathbb{Z}$, by

$$\tau^j(\mathbf{a}_0) := (\varepsilon_{j,0}, \dots, \varepsilon_{j,n-1}), \qquad j \in \mathbb{Z}. \tag{2.6}$$

We have the following

Lemma 2. *Let* $j_0 \in \mathbb{Z}$. *Then*

$$\varepsilon_{j,p} \equiv 2^{j-j_0}\varepsilon_{j_0,p} \pmod{n}, \tag{2.7}$$

for all $j > j_0$ *and* $p = 0, \dots, n-1$.

Proof. We first establish the relation:

$$\varepsilon_{j+1,p} \equiv 2\varepsilon_{j,p} \pmod{n}, \qquad j \leq -1, \ p = 0, \dots, n-1, \tag{2.8}$$

by induction. For $j = -1$, observe that for even p,

$$2\varepsilon_{-1,p} = 2\varepsilon_{0,p/2} \equiv p \pmod{n},$$

and that for odd p,

$$2\varepsilon_{-1,p} = 2\varepsilon_{0,[p/2]+(n+1)/2} = 2[p/2] + n + 1 \equiv p \pmod{n}.$$

Hence, (2.8) holds for $j = -1$. For $j < -1$, we have, for even p,

$$2\varepsilon_{j,p} = 2\varepsilon_{j+1,p/2} \equiv \varepsilon_{j+2,p/2} \pmod{n}$$

by applying the induction hypothesis. Since $\varepsilon_{j+2,p/2} = \varepsilon_{j+1,p}$, it follows that

$$2\varepsilon_{j,p} \equiv \varepsilon_{j+1,p} \pmod{n},$$

for even p. For odd p, we also have

$$\begin{aligned} 2\varepsilon_{j,p} &= 2\varepsilon_{j+1,[p/2]+(n+1)/2} \equiv \varepsilon_{j+2,[p/2]+(n+1)/2} \ (\mathrm{mod}\ n) \\ &\equiv \varepsilon_{j+1,p} \ (\mathrm{mod}\ n) \end{aligned}$$

again by applying the induction hypothesis. This establishes (2.8).

Of course, (2.7) is an immediate consequence of (2.8) for $0 \geq j > j_0$. On the other hand, if $j > 0 > j_0$, then we may obtain (2.7) by applying (2.5) in Lemma 1 as well. Finally, for $j > j_0 \geq 0$, then

$$\varepsilon_{j,p} \equiv 2^j \varepsilon_{0,p} \pmod{n} \equiv 2^{j-j_0} 2^{j_0} \varepsilon_{0,p} \pmod{n} \equiv 2^{j-j_0} \varepsilon_{j_0,p} \pmod{n}. \quad \square$$

Lemma 3. *Let j, $j_0 \in \mathbb{Z}$ with $j \geq j_0$ and $p = 0, \dots, n-1$. Then the two collections of functions*

$$\{2^j x - \varepsilon_{j,p}/n - k : k \in \mathbb{Z}\} \tag{2.9}$$

and

$$\{2^j(x - 2^{-j_0}\varepsilon_{j_0,p}/n) - k' : k' \in \mathbb{Z}\} \tag{2.10}$$

are identical.

Proof. Let $k \in \mathbb{Z}$. Since $\varepsilon_{j,k} \equiv 2^{j-j_0}\varepsilon_{j_0,p} \pmod{n}$, we have

$$2^j x - \frac{\varepsilon_{j,p}}{n} - k = 2^j x - \frac{2^{j-j_0}\varepsilon_{j_0,p}}{n} - k' = 2^j\left(x - \frac{2^{-j_0}\varepsilon_{j_0,p}}{n}\right) - k'$$

for some $k' \in \mathbb{Z}$. In addition, it is quite easy to see that the mapping $k \to k'$ is one-to-one. Hence, the two collections (2.9) and (2.0) are identical. $\square$

3. Proof of Theorem 1

Let n be a positive odd integer. We first decompose the collection of functions $\psi_{j,k/n}$, $j, k \in \mathbb{Z}$, into n disjoint subcollections $S_0, \dots, S_{n-1}$, where

$$S_p := \{2^{j/2}\psi(2^j x - \varepsilon_{j,p}/n - k) : j, k \in \mathbb{Z}\}. \tag{3.1}$$

Since S_0 is the set $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$, the assumption (1.3) can be expressed as

$$A\|f\|^2 \leq \sum_{g \in S_0} |\langle f, g\rangle|^2 \leq B\|f\|^2, \qquad f \in L^2. \tag{3.2}$$

Let $j_0 \in \mathbb{Z}$ and consider

$$\sigma_{j_0,p}(f) := \sum_{j \geq j_0}\sum_{k \in \mathbb{Z}} \left|\left\langle 2^{j/2}\psi\left(2^j \cdot \frac{\varepsilon_{j,p}}{n} - k\right), f\right\rangle\right|^2, \tag{3.3}$$

where $p = 0, \dots, n-1$, and $f \in L^2$. By Lemma 3, we see that

$$\begin{aligned}
\sigma_{j_0,p}(f) &= \sum_{j \geq j_0}\sum_{k \in \mathbb{Z}} \left|\left\langle 2^{j/2}\psi\left(2^j\left(\cdot - \frac{2^{-j_0}\varepsilon_{j_0,p}}{n}\right) - k\right), f\right\rangle\right|^2, \\
&= \sum_{j \geq j_0}\sum_{k \in \mathbb{Z}} \left|\left\langle 2^{j/2}\psi(2^j \cdot - k), f\left(\cdot + \frac{2^{-j_0}\varepsilon_{j_0,p}}{n}\right)\right\rangle\right|^2 \\
&\leq B\left\|f\left(\cdot + \frac{2^{-j_0}\varepsilon_{j_0,p}}{n}\right)\right\|^2 = B\|f\|^2.
\end{aligned}$$

Hence, for each $p = 1, \dots, n-1$, we have

$$\sum_{g \in S_p} |\langle f, g\rangle|^2 = \lim_{j_0 \to -\infty} \sigma_{j_0,p}(f) \leq B\|f\|^2.$$

Combining this with (3.3) yields

$$\sum_{j,k\in\mathbf{Z}} |\langle f, \psi_{j,k/n}\rangle|^2 = \sum_{p=0}^{n-1}\sum_{g\in S_p} |\langle f, g\rangle|^2 \le nB\|f\|^2. \tag{3.4}$$

To establish the lower bound in (1.7) we consider the class L_c^∞ of all a.e. bounded functions with compact support in $\mathbb{R}$. Since L_c^∞ is dense in L^2, it is sufficient to prove that the lower bound in (3.3) holds for all $f \in L_c^\infty$. Let $f \in L_c^\infty$ and suppose that

$$\operatorname{supp} f \subset [-L, L], \qquad L > 0. \tag{3.5}$$

Set

$$\boldsymbol{\Theta}_{j_0,p}(f) := \sum_{j\ge j_0}\sum_{k\in\mathbf{Z}} |\langle 2^{j/2}\psi(2^j\cdot -k), f_{j_0}^p\rangle|^2 \tag{3.6}$$

and

$$\Lambda_{j_0,p}(f) := \sum_{j<j_0}\sum_{k\in\mathbf{Z}} \left|\left\langle 2^{j/2}\psi\left(2^j\cdot - \frac{\varepsilon_{j,p}}{n} - k\right), f\right\rangle\right|^2, \tag{3.7}$$

where

$$f_{j_0}^p := f(x + 2^{-j_0}\varepsilon_{j_0,p}/n). \tag{3.8}$$

By (3.2), we have, for each $p = 1, \ldots, n-1$,

$$\sigma_{j_0,p}(f) + \boldsymbol{\Theta}_{j_0,p}(f) = \sum_{j,k\in\mathbf{Z}} |\langle \psi_{j,k}, f_{j_0}^p\rangle|^2 \ge A\|f_{j_0}^p\|^2 = A\|f\|^2.$$

This yields

$$\begin{aligned}\sum_{g\in S_p} |\langle f, g\rangle|^2 &= \sum_{j,k\in\mathbf{Z}} \left|\left\langle 2^{j/2}\psi\left(2^j\cdot - \frac{\varepsilon_{j,p}}{n} - k\right), f\right\rangle\right|^2 \\ &= \sigma_{j_0,p}(f) + \boldsymbol{\Theta}_{j_0,p}(f) - \boldsymbol{\Theta}_{j_0,p}(f) + \Lambda_{j_0,p}(f) \\ &\ge A\|f\|^2 - \boldsymbol{\Theta}_{j_0,p}(f).\end{aligned} \tag{3.9}$$

By introducing the notation

$$I_{j_0,p} := [-L - 2^{-j_0}\varepsilon_{j_0,p}/n, L - 2^{-j_0}\varepsilon_{j_0,p}/n],$$

it follows from (3.5) and (3.8) that

$$\operatorname{supp} f_{j_0}^p \subset I_{j_0,p}, \qquad p = 1, \ldots, n-1.$$

Hence, by the Cauchy inequality, we have

$$\begin{aligned}|\langle 2^{j/2}\psi(2^j\cdot -k), f_{j_0}^p\rangle|^2 &\le 2^{j+1}L\|f\|_\infty \int_{I_{j_0,p}} |\psi(2^jx-k)|^2\,dx \\ &= 2L\|f\|_\infty \int_{2^jI_{j_0,p}} |\psi(x-k)|^2\,dx.\end{aligned}$$

Let j_0, $J \in \mathbb{Z}$ be so chosen that $j_0 < -\log_2 nL$ and $J > \log_2(L+1)$. Then $I_{j_0,p} \subset (-\infty, 0)$, and hence

$$\Theta_{j_0,p}(f) \le 2L\|f\|_\infty \sum_{j_0-J\le j<j_0} \sum_{k\in\mathbb{Z}} \int_{k-2^jL-2^{-j_0+j}\varepsilon_{j_0,p}n^{-1}}^{k+2^jL-2^{-j_0+j}\varepsilon_{j_0,p}n^{-1}} |\psi(x)|^2\,dx$$
$$+ 2L\|f\|_\infty \sum_{k\in\mathbb{Z}} \int_{k-2^{-J}L-2^{-J}(n-1)n^{-1}}^{k} |\psi(x)|^2\,dx.$$

Let $\eta > 0$ be arbitrarily given. Since $\psi \in L^2$, there is some $\beta > 0$ such that $\int_{|x|\ge\beta} |\psi(x)|^2\,dx < \eta$. So, by setting $\psi_\beta := \psi\chi_{[-\beta,\beta]}$, $\beta > 0$, we have

$$\Theta_{j_0,p} \le 2L\|f\|_\infty \left(\eta + \sum_{j_0-J\le j<j_0} \sum_{k\in\mathbb{Z}} \int_{k-2^jL-2^{-j_0+j}\varepsilon_{j_0,p}n^{-1}}^{k+2^jL-2^{-j_0+j}\varepsilon_{j_0,p}n^{-1}} |\psi(x)|^2\,dx + \sum_{k\in\mathbb{Z}} \int_{k-2^{-J}L-2^{-J}(n-1)n^{-1}}^{k} |\psi_\beta(x)|^2\,dx\right), \tag{3.10}$$

where the last term on the right-hand side is smaller than η for any sufficiently large J. For such a fixed J, a $\gamma > 0$ can be chosen to yield

$$\int_{|x|\ge\gamma} |\psi(x)|^2\,dx \le \eta J^{-1}. \tag{3.11}$$

Hence, it follows from (3.10) that
(3.12)

$$\Theta_{j_0,p}(f) \le 2L\|f\|_\infty \left(3\eta + \sum_{j_0-J\le j\le j_0} \sum_{k\in\mathbb{Z}} \int_{k-2^jL-2^{-j_0+j}\varepsilon_{j_0,p}n^{-1}}^{k+2^jL-2^{-j_0+j}\varepsilon_{j_0,p}n^{-1}} |\psi_\gamma(x)|^2\,dx\right),$$

where (3.11) has been used to take care of $\psi - \psi_\gamma$. Since ψ_γ has compact support, the last term on the right-hand side of (3.12) tends to zero as $j_0 \to -\infty$, so that

$$\limsup_{j_0\to-\infty} \Theta_{j_0,p}(f) \le 6L\|f\|_\infty\eta, \qquad p = 1, \dots, n-1.$$

In view of (3.9) and (3.2), we have established the lower bound in (1.7).

Finally, to show that (1.8) does not necessarily hold for even $n > 0$, we consider two functions

$$f_1(x) = \psi_H(x+\tfrac{1}{2}) \quad \text{and} \quad f_2(x) = \chi_{[-1/2,1/2)}(x),$$

where $\psi_H(x) = \chi_{[0,1)}(x)\,\mathrm{sgn}(\frac{1}{2}-x)$ is the Haar function. Then for $\psi = \psi_H$ in (1.8), we have

$$\sum_{j,k\in\mathbb{Z}} |\langle f_1, \psi_{H;j,k/2}\rangle|^2 = 3\sum_{j,k\in\mathbb{Z}} |\langle f_2, \psi_{H;j,k/2}\rangle|^2 = \tfrac{9}{2},$$

while $\|f_1\| = \|f_2\| = 1$. Hence, $\{\psi_{H;j,k/2}\}$ cannot be a tight frame. $\square$

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