AN UNCERTAINTY PRINCIPLE ON HYPERBOLIC SPACE

LIMIN SUN

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ABSTRACT. In this paper, we establish an uncertainty principle on hyperbolic space \( H^n = SO_e(n, 1)/SO(n) \), which prohibits \( f \) from being confined to small neighborhoods around any point in \( H^n \) under certain assumptions on the Fourier transform \( \hat{f} \), where \( f \) is a normalized \( L^2 \) function on \( H^n \).

0. Introduction

For a function \( f \in L^2(R^n) \) with \( \|f\|_2 = 1 \), let \( \hat{f} \) be its Fourier transform. The classical Heisenberg uncertainty principle asserts that the product of the variances of the probability measures \( |f(x)|^2 \, dx \) and \( |\hat{f}(x)|^2 \, dx \) is larger than an absolute constant. Recently, Strichartz [3] gave a new generalization of the Heisenberg uncertainty principle, and posed a question: what conditions on the Fourier transform of a normalized \( L^2 \) function will force the variance of \( |f(x)|^2 \, dx \) to be large? As the author stated in his paper, this question makes sense in any context in which some sort of Fourier transform is defined. On the other hand, the Fourier transform on hyperbolic space \( H^n = SO_e(n, 1)/SO(n) \) can be defined following Helgason [1]. This naturally leads us to consider the above question on \( H^n \) in the present paper.

1. Preliminaries

Throughout this paper, we use \( \text{ch}, \text{sh}, \) and \( \text{th} \) as the simplified notations for \( \cosh, \sinh, \) and \( \tanh \) respectively.

Let \( G = SO_e(n, 1) \) be the identity component of the group \( SO(n, 1) = \{g \in GL(n + 1, R); g I_{n,1} g^t = I_{n,1}\} \), where \( I_{n,1} = \text{diag}(1, \ldots, 1, -1) \in GL(n + 1, R) \). The group \( K = SO(n) \) is naturally imbedded in \( SO_e(n, 1) \). It is well known that \( X = SO_e(n, 1)/SO(n) \) is a Riemannian symmetric space which is usually denoted by \( H^n \) and called hyperbolic space.

For further discussions, we take the open unit ball as a model for \( H^n \), i.e., \( X = \{x = (x_1, \ldots, x_n) \in R^n; |x| < 1\} \) with Riemannian structure \( ds^2 = 4(1 - |x|^2)^{-2}(dx_1^2 + \cdots + dx_n^2) \), where \( |\cdot| \) denotes the Euclidean norm in \( R^n \).

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Then the $G$-invariant measure on $X$ is given by
\[ dx = (2(1 - |x|^2)^{-1})^n \, dx_1 \cdots dx_n, \]
where $dx_1 \cdots dx_n$ is the Lebesgue measure on $\mathbb{R}^n$; and the $K$-invariant measure $db$ on the boundary $B$ ($= S^{n-1}$) is the standard surface element on $S^{n-1}$. The geodesic distance from $x \in X$ to the origin $o \in X$ is given by
\[
(1) \quad d(o, x) = \log \frac{1 + |x|}{1 - |x|}, \quad \text{i.e.,} \quad \text{th} \left( \frac{d(o, x)}{2} \right) = |x|.
\]

Let $(x, b)$ be the (signed) distance from $o$ to the horosphere passing through $x \in X$ and $b \in B$. A simple geometric observation shows that
\[
(2) \quad e^{(x, b)} = \frac{1 - |x|^2}{|x - b|^2}.
\]

The Fourier transform $\hat{f}$ of a function $f \in C_0^\infty (X)$ is defined by
\[
\hat{f}(\lambda, b) = \int_X e^{-i\lambda \cdot (x, b)} f(x) \, dx \quad \text{for} \quad \lambda \in C \quad \text{and} \quad b \in B.
\]

According to the general result by Helgason (cf. [2]), we have

**Theorem A.** The mapping $f \to \hat{f}$ extends to an isometry from $L^2(X, dx)$ onto $L^2(R_+ \times B, |C(r)|^{-2} \, dr \, db)$. Here, $C(\cdot)$ is Harish-Chandra's $c$-function.

## 2. Statement of the Result

Let $\{r_k\}$ be a series of real numbers satisfying the following conditions:
\[
(3) \quad \frac{d_0}{2} \leq r_1 \leq d_0 \leq 1, \quad 0 \leq r_{k+1} - r_k \leq d_0, \quad k = 1, 2, \ldots,
\]
\[
\lim_{k \to \infty} r_k = +\infty.
\]

Set
\[
(4) \quad E = \bigcup_{k=1}^\infty \{(r, b) \in R_+ \times B; r = r_k\}.
\]

A measure $m(r, b) \, dr \, db$ on $R_+ \times B$ can be induced on the submanifold $E$ in the following manner:
\[
\int_E h(r, b) m(r, b) \, dr \, db = \sum_{k=1}^\infty \int_E h(r_k, b) m(r_k, b) \, db.
\]

The main result of this paper is as follows.

**Theorem 1.** Let $r_k$ $(k = 1, 2, \ldots)$ and $E$ be given as in (3) and (4) respectively. Suppose that $f \in L^2(X, dx)$ with $\|f\|_2 = 1$. Fix an $\varepsilon \in (0, 1)$. Then for any sufficiently small $c > 0$, there exists a constant $c'$ such that
\[
\inf_{y \in X} \max_{\varepsilon} \left( \int_X d(x, y)^2 |f(x)|^2 \, dx, \int_X d(x, y)^{2+2\varepsilon} |f(x)|^2 \, dx \right) \geq c'd_0^{-2}
\]
provided that
\[
\int_E \left| \hat{f}(r, b) \right|^2 |C(r)|^2 \, dr \, db \leq c \cdot d_0^{-1}.
\]
Remark 1. Actually, we can choose \( c' = (2(1 - \sqrt{3nC^*c}))^2/(3n+2C^*C(\varepsilon)) \) for any \( 0 < c < (3nC^*)^{-1} \), where \( C^* = \text{ch}^2(2\pi) \cdot (n + 1)^{n-2} \) and \( C(\varepsilon) = 2(1 + 16/\varepsilon)^2 \).

3. SOME LEMMAS

For our purpose, we need an asymptotical estimation about the Harish-Chandra function.

**Proposition 1.** Let \( C(r) \) be the Harish-Chandra function in Theorem A. Then there exist constants \( \alpha \) and \( \beta \) dependent only on the dimension \( n \) such that \( \alpha r^{n-1} \leq |C(r)|^{-2} \leq \beta r^{n-1} \) for \( r \geq 1 \), and \( \alpha r^2 \leq |C(r)|^{-2} \leq \beta r^2 \) for \( 0 < r \leq 2 \).

**Proof.** We know from [5] that there exists a constant \( C_n \) dependent only on \( n \) such that, for \( r \in R \),

\[
|C(r)|^{-2} = C_n \left[ r^2 + \left( \frac{1}{2} \right)^2 \right] \cdots \left[ r^2 + \left( \frac{2m - 3}{2} \right)^2 \right] r \cdot \text{th}(\pi r), \quad n = 2m,
\]

and

\[
|C(r)|^{-2} = C_n \cdot r^2(r^2 + 1) \cdots (r^2 + (m - 1)^2), \quad n = 2m + 1.
\]

Hence, the conclusion follows.

**Remark 2.** In fact, one can choose \( \alpha = \pi C_n(\text{ch}^2(2\pi)2^{n-2})^{-1} \) and \( \beta = \pi C_n(4 + (n-3)^2/4)^{(n-2)/2} \). Note that \( \beta/\alpha \leq C^* \).

**Lemma 1.** Let \( \{r_k\} \) be given as in (3). If \( m_0 = 0 \) and \( m_k = \frac{1}{2}(r_k + r_{k+1}) \) for \( k \geq 1 \), then

\[
|C(r_k)|^2 \int_{m_{k-1}}^{m_k} |C(r)|^{-2} dr \leq (3nC^*)d_0.
\]

**Proof.** (I) The case \( k \geq 2 \). By the mean value theorem about integration, there exists \( t_k \in (m_{k-1}, m_k) \) such that

\[
|C(r_k)|^2 \int_{m_{k-1}}^{m_k} |C(r)|^{-2} dr = (m_k - m_{k-1})|C(r_k)/C(t_k)|^2.
\]

But, \( m_k - m_{k-1} = (r_{k+1} - r_k)/2 + (r_k - r_{k-1})/2 \leq d_0 \leq 1 \), and \( m_{k-1} \geq r_1 \geq d_0/2 \) for \( k \geq 2 \). Hence (note that \( r_k, t_k \in (m_{k-1}, m_k) \)),

\[
|r_k/t_k| \leq m_k/m_{k-1} = [1 + (m_k - m_{k-1})/m_{k-1}] \leq 3.
\]

Moreover, \( r_k, t_k \geq 1 \) if \( m_{k-1} \geq 1 \), and \( r_k, t_k < 2 \) if \( m_{k-1} < 1 \). Thus, one gets from (5), (6), and Proposition 1 that

\[
|C(r_k)|^2 \int_{m_{k-1}}^{m_k} |C(r)|^{-2} dr \leq C^* \left| \frac{r_k}{t_k} \right|^{n-1} \leq (3^{n-1}C^*)d_0 \quad \text{if} \quad m_{k-1} \geq 1,
\]

and

\[
|C(r_k)|^2 \int_{m_{k-1}}^{m_k} |C(r)|^{-2} dr \leq C^* \left| \frac{r_k}{t_k} \right|^{-1} \leq (9C^*)d_0 \quad \text{if} \quad m_{k-1} < 1.
\]

(II) The case \( k = 1 \). Since \( m_0 = 0 \) and \( d_0/2 \leq r_1 \leq m_1 \leq 3d_0/2 \). Applying Proposition 1, a simple calculation shows

\[
|C(r_1)|^2 \int_0^{m_1} |C(r)|^{-2} dr \leq C^*r_1^{-2} \int_0^{m_1} r^2 dr
\]

\[= C^*(m_1/r_1)^2(m_1/3) < (9C^*)d_0.
\]

The proof of Lemma 1 is complete.
Lemma 2. Let \( \{r_k\}, \{m_k\} \) be given as in Lemma 1. Write

\[
\begin{align*}
    a_k &= \int_{r_k}^{m_k} \int_{r_k}^{m_k} \left| \frac{C(r)}{C(t)} \right|^2 dt \, dr, \\
    b_k &= \int_{m_{k-1}}^{r_k} \int_{m_{k-1}}^{r_k} \left| \frac{C(r)}{C(t)} \right|^2 dt \, dr.
\end{align*}
\]

Then \( a_k \leq (3^nC^*/4)d_0^2 \) for \( k \geq 1 \), and \( b_k \leq (3^nC^*/4)d_0^2 \) for \( k \geq 2 \).

Proof. Again by the mean value theorem, there exist \( t_k \) and \( s_k \in (r_k, m_k) \) such that

\[
    a_k = (m_k - r_k)^2\left| \frac{C(s_k)}{C(t_k)} \right|^2 \leq (d_0/2)^2\left| \frac{C(s_k)}{C(t_k)} \right|^2.
\]

The remaining arguments are the same as those used in the proof of Lemma 1 (the case \( k \geq 2 \)). Since \( m_0 \) is not involved, we can deal with \( b_k \) in a similar way.

Lemma 3. Let \( \{r_k\}, \{m_k\} \) be given as in Lemma 1. If \( \hat{f} \) is the Fourier transform of a function \( f \) on \( X \), then for \( k \geq 1 \),

\[
\int_{m_{k-1}}^{m_k} \left( \int_{r_k}^{r} \left( \frac{\partial \hat{f}}{\partial s} \right)(s, b) \, ds \right)^2 \left| \frac{C(r)}{C(t)} \right|^{-2} \, dr \\
\leq \left( \frac{3^nC^*}{4} \right) d_0^2 \int_{m_{k-1}}^{m_k} \left\| \left( \frac{\partial \hat{f}}{\partial s} \right)(s, b) \right\|^2 \left| \frac{C(s)}{C(t)} \right|^{-2} \, ds.
\]

Proof. For convenience, write \( du(r) = \left| \frac{C(r)}{C(t)} \right|^{-2} \, dr \). The Hölder inequality simply yields

\[
\left( \int_{r_k}^{r} \left( \frac{\partial \hat{f}}{\partial r} \right)(s, b) \, ds \right)^2 \leq p(r, r_k) \int_{r_k}^{r} \left\| \left( \frac{\partial \hat{f}}{\partial r} \right)(s, b) \right\|^2 \, du(s),
\]

where \( p(r, r_k) = \int_{r_k}^{r} \left| \frac{C(s)}{C(t)} \right|^2 \, ds \). Changing the order of the integration, we get

\[
\begin{align*}
    \int_{m_{k-1}}^{m_k} \left( \int_{r_k}^{r} \left( \frac{\partial \hat{f}}{\partial r} \right)(s, b) \, ds \right)^2 \left| \frac{C(s)}{C(t)} \right|^{-2} \, ds \\
    &\leq \int_{m_{k-1}}^{m_k} \left\| \left( \frac{\partial \hat{f}}{\partial r} \right)(s, b) \right\|^2 du(s) \int_{m_{k-1}}^{m_k} -p(r, r_k) \, du(r) \\
    &\quad + \int_{r_k}^{m_k} \left\| \left( \frac{\partial \hat{f}}{\partial r} \right)(s, b) \right\|^2 du(s) \int_{s}^{m_k} p(r, r_k) \, du(r).
\end{align*}
\]

However, it follows from Lemma 2 that, for \( k \geq 1 \) and \( r_k \leq s \leq m_k \),

\[
\int_{s}^{m_k} p(r, r_k) \, du(r) \leq \int_{r_k}^{m_k} \int_{r_k}^{m_k} \left| \frac{C(s)}{C(r)} \right|^2 ds \, dr = a_k \leq \left( \frac{3^nC^*}{4} \right) d_0^2.
\]

Similarly, for \( k \geq 2 \) and \( m_{k-1} \leq s \leq r_k \),

\[
\int_{m_{k-1}}^{s} -p(r, r_k) \, du(r) \leq b_k \leq \left( \frac{3^nC^*}{4} \right) d_0^2.
\]
Finally, for $0 < r \leq r_1 \leq 1$,

$$-p(r, r_1) = \int_r^{r_1} |C(t)|^2 \, dt \leq \alpha^{-1} \int_r^{r_1} t^{-2} \, dt \leq (\alpha r)^{-1},$$

and hence, for $0 < s \leq r_1 \leq d_0$,

$$\int_0^s -p(r, r_1) \, du(r) \leq \frac{\beta}{\alpha} \int_0^s r \cdot dr \leq \left( \frac{C^*}{2} \right) \, d_0^2.$$

Now, Lemma 3 can be derived from (7)-(10) immediately.

Let $R(x) = \log[(1 + |x|^2)/(1 - |x|^2)]$. Then,

$$R(x) \leq \log[(1 + |x|^2)/(1 - |x|^2)] = d(o, x),$$

and

$$\log(1 - |x|^2)/|x - b|^2 = -\{R(x) + \log[|x - b|^2/(1 + |x|^2)]\}.\tag{12}$$

For $b = (b_1, \ldots, b_n) \in S^{n-1}$ and $l = (l_1, \ldots, l_n) \in \mathbb{Z}_+^n$, let $A_l(b) = (b_1)^{l_1} \cdots (b_n)^{l_n}$ and $|l| = l_1 + \cdots + l_n$. By Taylor's expansion,

$$\log |x - b|^2 = \log \left[ \frac{1 - 2x \cdot b}{1 + |x|^2} \right] = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{|l|=k} C_k^l A_l(b) Q_{k,l}(x),\tag{13}$$

where $C_k^l = k!/(l_1!) \cdots (l_n!)$.\break

Lemma 4. Let $\hat{f}$ be the Fourier transform of a function $f$ on $X$. Then, for any $0 < \varepsilon < 1$,

$$\int_{R^+ \times B} \left| \left( \frac{\partial \hat{f}}{\partial r} \right)(r, b) \right|^2 \cdot |C(r)|^{-2} \, dr \, db \leq C(\varepsilon) \max \left( \int_X d(o, x)^2 |f(x)|^2 \, dx, \int_X d(o, x)^{2+2\varepsilon} \sum |f(x)|^2 \, dx \right).$$

Proof. A direct calculation together with (12) and (13) shows

$$\left( \frac{\partial \hat{f}}{\partial r} \right)(r, b) = -i \int_X \left( \frac{\log(1 - |x|^2)}{|x - b|^2} \right) f(x) \left( \frac{1 - |x|^2}{|x - b|^2} \right)^{ir+(n-1)/2} \, dx$$

$$= -i(\tilde{Rf})(r, b) - i \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{|l|=k} C_k^l A_l(b) (\tilde{Q}_{k,l,f})(r, b).$$

But, $\sum_{|l|=k} C_k^l A_l(b)^2 = (b_1^2 + \cdots + b_n^2)^k = 1$. Hence, by Hölder's inequality we get that, for any $0 < \varepsilon < 1$,

$$\left( \frac{\partial \hat{f}}{\partial r} \right)^2 \leq 2 \left\{ (\tilde{Rf})^2 + C_1(\varepsilon) \sum_{k=1}^{\infty} \frac{(1 + \log k)^{1+\varepsilon}}{k} \sum_{|l|=k} C_k^l |\tilde{Q}_{k,l,f}|^2 \right\},$$

where $C_1(\varepsilon) = \sum_{k=1}^{\infty} 1/k(1 + \log k)^{1+\varepsilon}$. This fact together with Theorem A
shows that
\[
\int_{R^+ \times B} \left( \frac{\partial \tilde{f}}{\partial r} \right) (r, b) \cdot |C(r)|^{-2} \, dr \, db
\leq 2 \left\{ \int_X d(o, x)^2 |f(x)|^2 \, dx
+ C_1(e) \int_X \sum_{k=1}^{\infty} \frac{(1 + \log k)^{1+\epsilon}}{k} \sum_{|l|=k} C_l^2 |Q_{k,l}(x)|^2 f(x) |f(x)|^2 \, dx \right\}.
\]
However, a simple calculation yields that
\[
\sum_{k=1}^{\infty} \frac{(1 + \log k)^{1+\epsilon}}{k} \sum_{|l|=k} C_l^2 |Q_{k,l}(x)|^2 = \sum_{k=1}^{\infty} \frac{(1 + \log k)^{1+\epsilon}}{k} \left( \text{th}(d(o, x)) \right)^{2k}
\leq d(o, x)^2 + \sum_{k \geq 2} \frac{(1 + \log k)^{1+\epsilon}}{k} M_k^{2,\epsilon} d(o, x)^{2+2\epsilon},
\]
where \( M_{k,\epsilon} = \max_{r>0} \{ \text{th}(r)^k/r^{1+\epsilon} \} \). Obviously, \( M_{k,\epsilon} \leq 1 \) for all \( k \geq 2 \). If \( r_{k,\epsilon} \) is a stable point of \( \text{th}(r)^k/r^{1+\epsilon} \) then \( r_{k,\epsilon} \) must satisfy the equation \( 2r_k = (1 + \epsilon) \chi(2r) \). Hence, \( r_{k,\epsilon} \geq \frac{1}{2} \log(k-1) \) and \( M_{k,\epsilon} \leq [2/\log(k-1)]^{1+\epsilon} \) for \( k \geq 4 \). Finally, we get
\[
\int_{R^+ \times B} \left( \frac{\partial \tilde{f}}{\partial r} \right) (r, b)^2 |C(r)|^{-2} \, dr \, db
\leq 2(1 + C_1(e) + C_1(e)C_2(e))
\times \max \left( \int_X d(o, x)^2 |f(x)|^2 \, dx, \int_X d(o, x)^{2+2\epsilon} |f(x)|^2 \, dx \right),
\]
where \( C_2(e) = 6 + 4^{1+\epsilon} \sum_{k \geq 4} (1 + \log k)^{1+\epsilon} / k (\log(k-1))^{2+2\epsilon} \). It is easy to check that \( C_1(e) < 2/\epsilon \) and \( C_2(e) \leq 6 + 2^{1+\epsilon} \). This completes the proof.

4. PROOF OF THE THEOREM

First, we note the following fact that can be derived from the formulas in [1, pp. 197, 418].

Proposition 2. For a function \( f \) on \( X \), let \( f_g(x) = f(gx) \), \( g \in SO_k(n, 1) \). Then
(i) \( f_g(\lambda, b) = e^{(i\lambda+(n-1)/2)(g^o,g^b)} f(\lambda, g^b) \),
(ii) \( \int_B |\tilde{f}_g(\lambda, b)|^2 \, db = \int_B |\tilde{f}(\lambda, b)|^2 \, db \).

Now, we begin to prove Theorem 1. Suppose that \( f \) is a function on \( X \) such that \( \int_X (d(o, x)^{2+2\epsilon} |f(x)|^2 \, dx < \infty \) (if not, there is nothing to prove). Note the identity:
\[
\tilde{f}(r, b) = \tilde{f}(r_k, b) + \int_{r_k}^r \left( \frac{\partial \tilde{f}}{\partial r} \right) (s, b) \, ds.
\]
An application of Minkowski's inequality together with Lemmas 1 and 2 yields
\[
\left( \int_{m_{k-1}}^{m_k} |\tilde{f}(r, b)|^2 |C(r)|^{-2} dr \right)^{1/2} \\
\leq |\tilde{f}(r_k, b)||C(r_k)|^{-1} + \left( |C(r_k)|^2 \int_{m_{k-1}}^{m_k} |C(r)|^{-2} dr \right)^{1/2} \\
+ \left( \int_{m_{k-1}}^{m_k} \left( \int_{r_k}^r \left( \frac{\partial \tilde{f}}{\partial r} \right) (s, b) ds \right)^2 |C(r)|^{-2} dr \right)^{1/2} \\
\leq C_1 d_0^{1/2} |\tilde{f}(r_k b)| |C(r_k)|^{-1} \\
+ C_2 d_0 \left( \int_{m_{k-1}}^{m_k} \left( \frac{\partial \tilde{f}}{\partial r} \right) (s, b) \left| C(s) \right|^{-2} ds \right)^{1/2},
\]
where \( C_1 = 2C_2 = (3^n C^*)^{1/2} \). Another application of Minkowski's inequality together with Theorem A and Lemma 4 shows that
\[
1 = \left( \int_{R_+ \times B} |\tilde{f}(r, b)|^2 |C(r)|^{-2} dr db \right)^{1/2} \\
\leq C_1 d_0^{1/2} \left( \int_B \sum_{k=1}^{\infty} |\tilde{f}(r_k, b)|^2 |C(r_k)|^{-2} d b \right)^{1/2} \\
+ C_2 d_0 \left( \int_{R_+ \times B} \left( \frac{\partial \tilde{f}}{\partial r} \right) (s, b) \left| C(s) \right|^{-2} ds \right)^{1/2} \\
\leq C_1 d_0^{1/2} \left( \int_E |\tilde{f}(r, b)|^2 |C(r)|^{-2} dr db \right)^{1/2} \\
+ C_2 C(e)^{1/2} d_0 \max \left\{ \left( \int_X d(o, x)^2 |f(x)|^2 dx \right)^{1/2} \right\},
\]
(14)

Any \( y \in X \) can be written as \( y = go, g \in SO_e(n, 1) \). Proposition 2 implies that (14) remains true when \( d(o, x) \) is replaced by \( d(x, y) \). Hence, Theorem 1 is a direct consequence of (14).

Let \( I(X) \) be the space consisting of radial functions on \( X \). For \( f \in I(X) \), one can define its spherical transform by
\[
\hat{f}(\lambda) = \int_X f(x) \phi_{-\lambda}(x) dx, \quad \phi_{\lambda}(x) = \int_B e^{i\lambda (x^2-2e f(x))} db.
\]
Note that \( \hat{f}(\lambda, b) = \hat{f}(\lambda) \) for \( f \in I(X) \), and \( \int_B db = 1 \). Consequently,

**Theorem 2.** Let \( \{r_k\} \) be given as in (3) and \( r_{-k} = -r_k, k = 1, 2, \ldots \). Suppose that \( f \in I(X) \cap L^2(X) \) with \( \|f\|_2 = 1 \). Fix an \( \epsilon \in (0, 1) \). Then for any
sufficiently small $c > 0$, there exists constant $c'$ such that
\[ \inf_{y \in X} \max_{\alpha \in \mathbb{R}} \left( \int_X d(x, y)^2 |f(x)|^2 dx, \int_X d(x, y)^{2+2\alpha} |f(x)|^2 dx \right) \geq c'd_0^{-2} \]
provided that $\sum_{k=-\infty}^{+\infty} |c(k)|^{-2} |\hat{f}(r_k)|^2 \leq c \cdot d_0^{-1}$.

5. A FURTHER RESULT

Obviously, for a radial function $f$ on $X$, one should study its confinement to some annular neighborhood $N_\delta(R) = \{ x \in X ; |d(o, x) - R| < \delta \}$; i.e., we hope to get the lower bound for
\[ \int_X |d(o, x) - R|^2 |f(x)|^2 dx, \quad R > 0, \]
under suitable conditions on the Fourier transform of $f$. Here, we present an example to explain this point.

Let us consider $X$ for $n = 3$. In this case, $|C(\lambda)|^{-2} = (2\pi)^{-3}\lambda^2$ for $\lambda > 0$, and
\[ \phi_\lambda(x) = 2\pi \int_0^\infty \left( ch r - sh r \cdot cos \theta \right)^{-1+\lambda} (sin \theta)^{\alpha-2} d\theta \]
\[ = -4\pi sin(\lambda r)/\lambda sh(r) \quad \text{for} \ d(o, x) = r. \]
Calculating under the geodesic coordinates, we have $dx = (sh r)^2 dr dw$, where $dw$ is the standard surface element on $S^2$. Hence, each radial function $f(x) \in I(X) \cap L^2(X, dx)$ corresponds to a function $F_f(r) \in L^2_{odd}(R, dr)$ such that $f(x) = F_f(r)/sh(r)$ for $r = d(o, x)$, where $L^2_{odd}(R, dr)$ denotes the space consisting of odd functions which are square integrable on $R$.

**Proposition 3.** The mapping: $f \rightarrow F_f$ is an isometry from $I(X) \cap L^2(X, dx)$ onto $L^2_{odd}(R, dr/2)$. Moreover, $\hat{f}(\lambda) = -i(2\pi)^{-1/2} |C(\lambda)| \cdot \hat{F}_f(\lambda)$ for $\lambda > 0$, where $\hat{f}$ is the spherical transform of $f$ and $\hat{F}_f$ is the Euclidean Fourier transform of $F_f$.

**Proof.** The first assertion is obvious. In the present case, $|C(\lambda)|^{-2} = (2\pi)^{-3}\lambda^2$ for $\lambda > 0$. By (15) a direct calculation shows
\[ \hat{f}(\lambda) = \int_X f(x) \phi_\lambda(x) dx = \frac{2\pi}{\lambda} \int_{-\infty}^{+\infty} F_f(r)e^{-i\lambda r} dr \]
\[ = i(2\pi)^{-1/2} |C(\lambda)| \hat{F}_f(\lambda), \quad \lambda > 0. \]

**Theorem 3.** Let $\{r_k\}$ be given as in (3). Suppose that $f \in I(X) \cap L^2(X, dx)$ with $\|f\|_2 = 1$. Then for any $0 < \delta < 1$,
\[ \inf_{y \in X, R \geq 0} \int_X |d(x, y) - R|^2 |f(x)|^2 dx \geq \frac{8\delta^2}{d_0^2}, \]
provided that $\sum_{k=1}^{\infty} |\hat{f}(r_k)|^2 |C(r_k)|^{-2} \leq (1 - \delta^2)/2\pi d_0$. Here, $X$ is three dimensional hyperbolic space.

**Proof.** Since the measure $dx$ is invariant, (16) is equivalent to
\[ \inf_{R \geq 0} \int_X |d(o, x) - R|^2 |f(x)|^2 dx \geq \frac{8\delta^2}{d_0^2}. \]
By Proposition 3, we need only show
\[
\inf_{R \geq 0} \frac{1}{2} \int_{-\infty}^{\infty} |r - R|^2 |F_f(r)|^2 \, dr \geq \frac{8\delta^2}{d_0^2}.
\]
Again by Proposition 3, one sees that
\[
\left( \frac{1}{2} \sum_{k = -\infty}^{\infty} |\hat{F}_f(r_k)|^2 \right)^{1/2} = \left( \sum_{k = 1}^{\infty} |\hat{F}_f(r_k)|^2 \right)^{1/2}
\leq \left( 2\pi \sum_{k = 1}^{\infty} |\hat{f}(r_k)|^2 |C(r_k)|^{-2} \right)^{1/2} \leq \frac{1 - \delta}{d_0^{1/2}}.
\]
The conclusion follows from the result by Strichartz in [3, p. 98].

**Remark 3.** It would be interesting to know whether the conclusion in Theorem 3 is valid in general hyperbolic space. To deal with this problem will require a new technique since the one we used relies on the particular expression of the spherical function.

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**DEPARTMENT OF MATHEMATICS, HANGZHOU UNIVERSITY, HANGZHOU, ZHEJIANG, 310028, PEOPLES REPUBLIC OF CHINA**