PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 2, June 1994

A NON-ITERATIVE 2-ADIC STATEMENT OF THE $3N+1$ CONJECTURE

DANIEL J. BERNSTEIN

(Communicated by William Adams)

ABSTRACT. Associated with the $3N+1$ problem is a permutation Φ of the 2-adic integers. The $3N+1$ conjecture is equivalent to the conjecture that $3Q$ is an integer if $\Phi(Q)$ is a positive integer. We state a new definition of Φ. To wit: Q and $N = \Phi(Q)$ are linked by the equations $Q = 2^{d_0} + 2^{d_1} + \cdots$ and $N = (-1/3)2^{d_0} + (-1/9)2^{d_1} + (-1/27)2^{d_2} + \cdots$ with $0 \le d_0 < d_1 < \cdots$. We list four applications of this definition.

DEFINITION AND CONJECTURE

We recall that the 2-adic integers $\mathbf{Z}_2$ may be defined as binary expansions which are allowed to extend infinitely far to the left [3, Exercise 4.1–31]: for instance, $1/3 = (\ldots 0101011)_2 \in \mathbf{Z}_2$. Fix odd 2-adic integers $u_0, u_1, \ldots \in 1 + 2\mathbf{Z}_2$. For any increasing, finite or infinite, sequence $0 \le d_0 < d_1 < \cdots$ of nonnegative integers, the sum $\sum u_i 2^{d_i}$ converges to some 2-adic integer S. In fact this map from increasing sequences to sums is one-to-one and onto all of $\mathbf{Z}_2$. We construct its inverse. Given $S \in \mathbf{Z}_2$ we set d_0 to the first bit position in S, d_1 to the first bit position in $S - u_0 2^{d_0}$, and so on. If at any point $S - u_0 2^{d_0} - \cdots - u_n 2^{d_n}$ is zero, we stop, and the sequence is finite.

As an example, the expansion

$$Q = 2^{d_0} + 2^{d_1} + \cdots \tag{1}$$

is a bijection between $Q \in \mathbf{Z}_2$ and increasing sequences $d = \langle d_0, d_1, \ldots \rangle$. Finite sequences d correspond to nonnegative integers Q. (In particular the empty sequence corresponds to $Q = 0$.) Similarly, the expansion

$$N = \frac{-1}{3}2^{d_0} + \frac{-1}{9}2^{d_1} + \frac{-1}{27}2^{d_2} + \cdots \tag{2}$$

is another bijection. Together (1) and (2) determine a bijection between all

Received by the editors July 13, 1992 and, in revised form, September 30, 1992.
1991 *Mathematics Subject Classification*. Primary 11D72.
The author was supported in part by a National Science Foundation Graduate Fellowship. He would also like to thank Bellcore for the use of their facilities in preparing this document.

$N \in \mathbf{Z}_2$ and all $Q \in \mathbf{Z}_2$. We write $N = \Phi(Q)$. For instance,

$$(3) \qquad \Phi(-\frac{1}{3}) = \Phi(2^0 + 2^2 + 2^4 + \cdots) = \frac{-1}{3}2^0 + \frac{-1}{9}2^2 + \frac{-1}{27}2^4 + \cdots = 1$$

by elementary geometric series manipulations. So $1 \in \Phi((1/3)\mathbf{Z})$.

Conjecture. *The set* $\mathbf{Z}^+$ *of positive integers is contained in* $\Phi((1/3)\mathbf{Z})$.

Connection with the $3N+1$ problem

We define two functions $H(Q)$ and $C(N)$. If Q is even we set $H(Q) = Q/2$; otherwise we set $H(Q) = Q - 1$. If N is even we set $C(N) = N/2$; otherwise we set $C(N) = 3N + 1$.

Theorem 1. $C(\Phi(Q)) = \Phi(H(Q))$.

Proof. We define d as in (1). If Q is even then $d_0 > 0$ (or d has length 0) and

$$C(\Phi(Q)) = \frac{-1}{3}2^{d_0-1} + \frac{-1}{9}2^{d_1-1} + \cdots = \Phi(2^{d_0-1} + 2^{d_1-1} + \cdots) = \Phi(Q/2).$$

If Q is odd then $d_0 = 0$ and

$$\begin{aligned} C(\Phi(Q)) &= 1 + 3(\frac{-1}{3} + \frac{-1}{9}2^{d_1} + \frac{-1}{27}2^{d_2} + \cdots) = \frac{-1}{3}2^{d_1} + \frac{-1}{9}2^{d_2} + \cdots \\ &= \Phi(2^{d_1} + 2^{d_2} + \cdots) = \Phi(Q - 2^{d_0}) = \Phi(Q - 1) \end{aligned}$$

as desired. □

The $3N+1$ conjecture [4] states that, for any positive integer $N \in \mathbf{Z}^+$, some iterate $C^k(N)$ equals 1. This implies our conjecture:

Theorem 2. *If* $C^k(N) = 1$ *then* $N \in \Phi((1/3)\mathbf{Z})$.

Proof. Set $Q = \Phi^{-1}(N)$. Now $C^k(\Phi(Q)) = 1$ so $\Phi(H^k(Q)) = 1$ so $H^k(Q) = \Phi^{-1}(1) = -1/3 \in (1/3)\mathbf{Z}$. If $H(x) \in (1/3)\mathbf{Z}$ then $x \in (1/3)\mathbf{Z}$, so by induction $Q \in (1/3)\mathbf{Z}$ as desired. □

The converse is also true: our conjecture implies the $3N + 1$ conjecture.

Theorem 3. *If* $N \in \mathbf{Z}^+$ *and* $N \in \Phi((1/3)\mathbf{Z})$ *then* $C^k(N) = 1$ *for some* k.

Proof. Again set $Q = \Phi^{-1}(N)$ and define d by (1) and (2). We have $3Q \in \mathbf{Z}$. Notice first that Q cannot be an integer. For if $Q \in \mathbf{Z}$ then either $Q = 0$, in which case $N = 0$; or Q is positive, in which case d is finite and N is a negative rational number by (2); or Q is negative, in which case $d_{i+1} = d_i + 1$ for all large i and (2) again converges to a negative rational number.

So Q is $1/3$ away from an integer. Thus the d_i's eventually fall into the pattern $d_{i+1} = d_i + 2$, say for $i \geq m$. Define a map c on d corresponding to the action of C on N. Notice that c acts on d by subtracting 1 from each element, if $d_0 > 0$; or by shifting d to the left, if $d_0 = 0$. So

$$c^{d_m+m}(\langle d_0, \ldots, d_{m-1}, d_m, d_m + 2, d_m + 4, \ldots\rangle) = \langle 0, 2, 4, \ldots\rangle,$$

and $C^{d_m+m}(N) = 1$. □

Hence our conjecture about the 2-adic expansions (1) and (2) is equivalent to the $3N+1$ conjecture. Does this throw any light on the latter? Our map Φ is exactly the inverse of Q_∞ in [4]. (This can alternatively be derived from [5, Lemma 4].) Our Theorem 1, that Φ conjugates H to C, is equivalent to [1, Theorem 1]. What is new here is the expansion (2). It gives an explicit formula for $\Phi = Q_\infty^{-1}$. We have therefore answered affirmatively the final question in [4].

Applications

Our result has several immediate applications. First, say $Q \in \mathbf{Q} \cap \mathbf{Z}_2$ is rational. Then either d is finite (say of length μ) or $d_{m+\lambda} = d_m + X$ for all sufficiently large m (say $m \geq \mu$) and some fixed λ and X. In the first case $3^\mu N$ is an integer. In the second case

$$-3^\mu(3^\lambda - 2^X)N = (3^\lambda - 2^X)(3^{\mu-1}2^{d_0} + \cdots + 3^0 2^{d_{\mu-1}}) + (3^{\lambda-1}2^{d_\mu} + \cdots + 3^0 2^{d_{\mu+\lambda-1}})$$

by (2). So in either case N is rational.

Corollary 1. $\Phi(\mathbf{Q} \cap \mathbf{Z}_2) \subseteq \mathbf{Q} \cap \mathbf{Z}_2$.

The "Periodicity Conjecture" from [4] states that $\Phi(\mathbf{Q} \cap \mathbf{Z}_2) = \mathbf{Q} \cap \mathbf{Z}_2$. We have shown half of this.

Second, in our development of Φ we noted that Φ is a bijection from $\mathbf{Z}_2$ onto itself, i.e., a permutation of $\mathbf{Z}_2$. In fact we see from (1) and (2) that Φ is a homeomorphism under the topology induced by the usual 2-adic metric (see [4, §2.8]). So [4, Theorem L] follows immediately.

Third, from (1) and (2) we see that the function $Q_k(N) = \Phi^{-1}(N) \bmod 2^k$ depends only on the equivalence class $N \bmod 2^k$. This is the first half of [4, Theorem B]. The induced function $\overline{Q}_k$ on equivalence classes is a permutation because Φ is. By induction the cycles of $\overline{Q}_k$ are of length dividing 2^k. Indeed, any cycle of $\overline{Q}_k$ of length r gives rise to either one cycle of $\overline{Q}_{k+1}$ of length $2r$ or two cycles of $\overline{Q}_{k+1}$ of length r. This is the second half of [4, Theorem B].

Finally, we give a short proof of the following theorem of Müller [5].

Corollary 2. Φ *is nowhere differentiable.*

Proof. If d is infinite then for any $k \geq 0$

$$\frac{\Phi(Q) - \Phi(Q - 2^{d_k})}{2^{d_k}} \equiv (-1)^k \pmod 4$$

by routine computation from (2). So as $k \to \infty$ the difference ratio does not converge. If d is finite, say of length m, then for $e + f > e > d_{m-1}$

$$\frac{\Phi(Q) - \Phi(Q + 2^e + 2^{e+f})}{2^e + 2^{e+f}} = \frac{1}{3^{m+2}} \frac{3 + 2^f}{1 + 2^f}$$

again by routine computation; and the latter quantity is different mod 8 for $f = 1, 2$. So as $e \to \infty$ the difference ratio does not converge. Hence both Φ and its inverse are nowhere differentiable. □

Note that our approach generalizes to the "$AN + B$ problem" [2, 6]. In this generalization A and B are odd, and $N = \Phi_{A,B}(Q) = \sum(-B/A^{i+1})2^{d_i}$. This homeomorphism $\Phi_{A,B}$ conjugates H to $C_{A,B}$, where $C_{A,B}(N)$ equals $N/2$ for N even and $AN + B$ for N odd.

Acknowledgment

The author would like to thank Jeffrey C. Lagarias and an anonymous referee for their helpful suggestions.

References

1. Ethan Akin, $3x + 1$, unpublished manuscript.
2. R. E. Crandall, *On the* $3x + 1$ *problem*, Math. Comp. **32** (1978), 1281–1292.
3. Donald E. Knuth, *The art of computer programming, volume* 2: *Seminumerical algorithms*, 2nd ed., Addison-Wesley, Reading, MA, 1981.
4. Jeffrey C. Lagarias, *The* $3x + 1$ *problem and its generalizations*, Amer. Math. Monthly **92** (1985), 3–23.
5. Helmut Müller, *Das* $3n + 1$ *Problem*, Mitt. Math. Ges. Hamburg **12** (1991), 231–251.
6. Ray P. Steiner, *On the* $Qx + 1$ *problem,* Q *odd.* II, Fibonacci Quart. **19** (1981), 293–296.

5 Brewster Lane, Bellport, New York 11713
E-mail address: djb@silverton.berkeley.edu