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INCOMPATIBILITY OF COMPACT PERTURBATIONS WITH THE SZ. NAGY-FOIAŞ FUNCTIONAL CALCULUS

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ABSTRACT. For every absolutely continuous contraction T with spectrum on the unit circle, we exhibit an H^∞ function h and a sequence of operators T_n which are unitarily equivalent to T and differ from T by a sequence of compact operators converging to 0 in norm such that $h(T_n)$ is never a compact perturbation of $h(T)$. When T is diagonal, it can also be arranged that $T-T_n$ is trace class, and T_n commutes with T.

RÉSUMÉ. Pour toute contraction absolument continue T dont le spectre rencontre le cercle unité, il existe une fonction h de H^∞ et une suite T_n d'opérateurs unitairement equivalents à T telle que $T-T_n$ soit compact et convergent en norme vers 0, mais $h(T_n)-h(T)$ soit non compact pour tout n. Dans le cas où T est diagonal, la suite T_n vérifie en plus $T-T_n$ est un opérateur à trace, et T_n commute avec T.

In this note, we show that the Sz. Nagy-Foiaş functional calculus is incompatible with compact perturbations in a very strong sense. This should not be too surprising, as the algebra H^∞ is not separable, and neither is the algebra generated by an (absolutely continuous) contraction under this functional calculus whenever the spectrum meets the unit circle (see the lemma below). Thus one should not expect the good behaviour under the norm required to preserve compact perturbations. Nevertheless, it may be a bit surprising that this fails for *all* such contractions. Moreover, we perturb the operator to another which is unitarily equivalent and at the same time, a small compact perturbation. The key to this construction is Voiculescu's double commutant theorem for separable subalgebras of the Calkin algebra, which is a corollary of his famous generalization of the Weyl-von Neumann theorem [5]. In the case of diagonal operators, we can do even better by obtaining perturbations by commuting, trace class operators of small trace norm. In this case, the argument is more elementary.

Let $\mathscr{H}$ denote a separable Hilbert space. The spaces $\mathscr{L}(\mathscr{H})$, $\mathscr{K}=\mathscr{K}(\mathscr{H})$, and $\mathscr{Q}(\mathscr{H})=\mathscr{L}(\mathscr{H})/\mathscr{K}$ are, respectively, the space of bounded linear operators on $\mathscr{H}$, the ideal of compact operators, and the Calkin algebra. Let π denote the canonical projection of $\mathscr{L}(\mathscr{H})$ onto $\mathscr{Q}(\mathscr{H})$.

By H^∞, we mean the algebra of bounded analytic functions on the unit disc $\mathbf{D}$. A subset Λ of $\mathbf{D}$ is called interpolating if the restriction map of H^∞

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to Λ, $(h \to h|\Lambda)$, is onto $l^\infty(\Lambda)$. Any sequence converging to the boundary sufficiently fast has this property. We have no need of Carleson's difficult characterization of these sets.

A contraction (i.e., $T \in \mathscr{L}(\mathscr{H})$ such that $\|T\| \leq 1$) is absolutely continuous if the spectral measure of its unitary part is absolutely continuous with respect to Lebesgue measure. In this case, there is an H^∞ functional calculus [4]. We denote by $H^\infty(T)$ the set $\{h(T) | h \in H^\infty\}$. Let $\mathscr{C}_1$ denote the ideal of trace class operators with the trace norm $\|C\|_1 = \operatorname{tr}(|C|)$ where $|C| = (C^*C)^{1/2}$ and tr is the trace. We will use the notations $[A, B] = AB - BA$, and $A \cong B$ means unitary equivalence.

The first lemma establishes the desired nonseparability of $H^\infty(T)$. The argument is an easy application of a clever function theoretic construction of [1] which shows that if T is a completely nonunitary contraction with connected spectrum meeting the unit circle, then there is an H^∞ function f so that $\|f\| = 1$ and $\sigma(f(T)) = \overline{\mathbf{D}}$.

Lemma 1. *If T is an absolutely continuous contradiction with* $\operatorname{spr}(T) = 1$, *then $H^\infty(T)$ is not separable.*

Proof. We may suppose that $1 \in \sigma(T)$. If 1 is an isolated point of the spectrum, then T has an invariant subspace $\mathscr{N}$ so that $\sigma(T|_{\mathscr{N}}) = \{1\}$. The map taking $h(T)$ to $h(T|_{\mathscr{N}})$ is contractive. Thus an application of the result just cited shows that H^∞ is a quotient of $H^\infty(T)$, and thus this is not separable.

Otherwise, the operator $X = (I+T)/2 = g(T)$ is a contradiction in $H^\infty(T)$ with spectrum contained in $\mathbf{D} \cup \{1\}$ such that 1 is not isolated. Hence it contains a sequence of points in the disc converging to 1. By dropping to a subsequence, it may be assumed that this is an interpolating Blaschke sequence, say Λ. But then for $h \in H^\infty$, one has

$$\|h \circ g(T)\| \geq \operatorname{spr}(h(X)) \geq \|h|_\Lambda\|.$$

It follows that the map taking $h \circ g(T)$ to $h|_\Lambda$ is a continuous map onto l^∞. Hence $H^\infty(T)$ is not separable. □

Theorem 2. *Suppose that T is an absolutely continuous contraction with spectral radius* 1. *Let $h \in H^\infty$ be any function such that $h(T)$ does not belong to $C^*(T) + \mathscr{K}$. Then there is a selfadjoint operator A so that the operators $T_t = e^{itA}Te^{-itA}$ converge to T as $t \to 0$, $T_t - T$ is compact for all $t \in \mathbf{R}$, and $h(T_t) - h(T)$ is never compact for $0 < |t| < \pi$.*

Proof. First notice that such h exist because, by the previous lemma, $H^\infty(T)$ is not separable, yet $C^*(T) + \mathscr{K}$ clearly is separable.

By Voiculescu's double commutant theorem, the separable C^*-subalgebra $C^*(\pi T)$ of the Calkin algebra is equal to its own double commutant. Thus since $\pi h(T)$ is not in this algebra, there is a selfadjoint element πA so that $[A, T]$ is compact, but $[A, h(T)]$ is not. Normalize A so that it is norm 1. Clearly, the unitaries $U_t = e^{itA}$ also commute with T modulo the compacts, but they cannot commute with $h(T)$ modulo $\mathscr{K}$ for any $|t| < \pi$.

Define $T_t = U_t T U_t^*$ for $|t| < \pi$. It is immediate that $T_t - T$ is compact, and $\lim_{t\to 0} \|T_t - T\| = 0$. Also, $h(T_t) - h(T) = [U_t, h(T)]U_t^*$ is never compact for $|t| < \pi$. □

Remark 3. When $r = \operatorname{spr}(T) < 1$, the H^∞ functional calculus is absolutely convergent (using the Taylor series). Thus in this case, $H^\infty(T)$ is contained

in the norm closed algebra generated by T. If K is any compact operator such that $T+K$ is an absolutely continuous contraction, then $\operatorname{spr}(T+K) < 1$. To see this, notice that any point λ in the spectrum of $T+K$ with $|\lambda| > r$ must be an eigenvalue of finite multiplicity. From the absolute continuity, it follows that $|\lambda| < 1$. There are only finitely many points λ in $\sigma(T+K)$ with $|\lambda| > (1+r)/2$. So the claim follows. In this case, it is easy to see that $h(T+K) - h(T)$ is compact for every $h \in H^\infty$ [3].

The case of diagonal operators with point spectrum in the disc is amenable to a finer analysis. We quote the following result from [3].

Theorem 4 (Esterle-Zarouf). *Suppose that* $T = \operatorname{diag}(\tau_n)$ *and* $S = \operatorname{diag}(\sigma_n)$ *are diagonal operators with respect to the same basis, and both operators have their point spectrum contained in the open unit disc. Then* $h(S) - h(T)$ *is compact for every* $h \in H^\infty$ *if and only if*

$$\lim_{n\to\infty} \frac{\sigma_n - \tau_n}{1 - |\tau_n|} = 0.$$

Using this result, we obtain the following refinement of our previous theorem.

Theorem 5. *Suppose that* $T = \operatorname{diag}(\tau_n)$ *with* $|\tau_n| < 1 = \sup_n |\tau_n|$. *Then there exists an* H^∞ *function* h *and operators* T_n *so that for all* $n \geq 1$,

(i) $T_n \cong T$,
(ii) $T_n T = T T_n$,
(iii) $T_n - T \in \mathscr{C}_1$,
(iv) $\lim_n \|T_n - T\|_1 = 0$, *and*
(v) $h(T_n) - h(T)$ *is not compact.*

Proof. There is a subsequence n_k so that $\lim_k \tau_{n_k} = \lambda$ and $|\lambda - \tau_{n_{k+1}}| < 2^{-k}(|\lambda - \tau_{n_k}|)$. It clearly suffices to prove the result in the case when $n_k = k$ and $\lambda = 1$. So we now make that assumption.

Consider the permutations π_k of the positive integers $\mathbf{N}$ given by $\pi_k(2j-1) = 2j$ and $\pi_k(2j) = 2j-1$ for $j > k$, and $\pi_k(j) = j$ for $j \leq 2k$. Let $T_k = \operatorname{diag}(\tau_{\pi_k(j)})$. From this definition, conditions (i) and (ii) are obvious. And

$$\|T_k - T\|_1 = \sum_{j>k} 2|\tau_{2j-1} - \tau_{2j}| < 2 \sum_{j>2k} |1 - \tau_j|.$$

So (iii) and (iv) follow.

However, one easily sees that

$$\overline{\lim_{k\to\infty}} \frac{|\tau_{2k-1} - \tau_{2k}|}{1 - |\tau_{2k}|} = 1.$$

Hence, by the previous theorem, there is a function h so that $h(T_1) - h(T)$ is not compact. Since $h(\operatorname{diag}(\sigma_n)) = \operatorname{diag}(h(\sigma_n))$, it is clear that $h(T_1) - h(T_k)$ is finite rank for all k. Thus, $h(T_k) - h(T)$ is never compact. □

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