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BAIRE PARADOXICAL DECOMPOSITIONS NEED AT LEAST SIX PIECES

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ABSTRACT. We show that in certain cases paradoxical decompositions of compact metric spaces using sets (or even $[0, 1]$-valued functions) with the property of Baire modulo meager sets need more pieces than paradoxical decompositions with unrestricted pieces. In particular, any Baire paradoxical decomposition of the sphere S^2 using isometries needs at least six pieces.

Let G be a group of isometries of a compact metric space (X, d). Denote by $\mathscr{B}(X)$ the additive semigroup of real-valued bounded positive functions on X that have the Baire property (see [2] for this and related terminology, e.g., first category, Baire category theorem, etc.). If φ and ψ are two elements of $\mathscr{B}(X)$, $\varphi \geq \psi$ (B-a.e.) will mean that the set of all x such that $\varphi(x) \not\geq \psi(x)$ is of first category and similarly for $\varphi = \psi$ (B-a.e.). There is a canonical action of G on $\mathscr{B}(X)$ defined by $(g\varphi)(x) = \varphi(g^{-1}x)$. Two elements φ and ψ of $\mathscr{B}(X)$ are said to be (*continuously*) *G-equidecomposable* (see [4]) when there are n in $\mathbb{N}$, $\varphi_1, \dots, \varphi_n$ in $\mathscr{B}(X)$, and $g_1, \dots, g_n$ in G such that

$$\varphi = \sum_{i=1}^{n} \varphi_i \quad \text{(B-a.e.)} \qquad \text{and} \qquad \psi = \sum_{i=1}^{n} g_i\varphi_i \quad \text{(B-a.e.)}.$$

When the φ_i are restricted to be $\{0, 1\}$-valued functions, this is the usual notion of Baire equidecomposability, which we shall call *discrete*, as opposed to continuous (see [4]). In general, continuous equidecomposability is weaker than discrete equidecomposability. We will say that φ is *Baire paradoxical* (with respect to the group G) when φ and 2φ are equidecomposable in $\mathscr{B}(X)$; we will emphasize "continuous" or "discrete" if the context does not make it clear. If $Y \subseteq X$, we will identify Y with its characteristic function $\mathbf{1}_Y$.

Lemma. *Let φ in $\mathscr{B}(X)$ and g in G be such that $g\varphi \geq \varphi$ (B-a.e.). Then $g\varphi = \varphi$ (B-a.e.).*

Proof. There is a comeager subset Y of X such that the restriction $\varphi|_Y$ is continuous (see [2]); furthermore, one can suppose without loss of generality that $gY = Y$ and that $(\forall x \in Y)(\varphi(g^{-1}x) \geq \varphi(x))$. We prove that for all x

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in Y, we have in fact $\varphi(g^{-1}x) = \varphi(x)$. For, let $(g^{-n_k}x)_{k\in\mathbb{N}}$ be a convergent subsequence of $(g^{-n}x)_{n\in\mathbb{N}}$ and put $m_k = n_{k+1} - n_k$ so that $m_k > 0$. Since g is an isometry, we have $x = \lim_{k\to\infty} g^{-m_k}x$; since $\varphi|_Y$ is continuous, it follows that $\varphi(x) = \lim_{k\to\infty} \varphi(g^{-m_k}x)$. But for all k we have $\varphi(g^{-m_k}x) \geq \varphi(g^{-1}x) \geq \varphi(x)$, whence $\varphi(g^{-1}x) = \varphi(x)$. □

As a possible application, we get, for example, the following

Corollary. *Let* $m, n \in \mathbb{N}$, $g_i \in G$, *and* $\varphi_i \in \mathscr{B}(X)$ $(1 \leq i \leq m+n)$ *such that*

$$1_X = \sum_{i=1}^{m+n} \varphi_i = \sum_{i=1}^{m} g_i\varphi_i = \sum_{i=m+1}^{m+n} g_i\varphi_i \quad \text{(B-a.e.).}$$

Then $m \geq 3$ *and* $n \geq 3$.

Thus, if X is Baire paradoxical, then any Baire-paradoxical decomposition of X uses at least six pieces; of course, this works as well for continuous as for discrete equidecomposability. Furthermore, this has been recently shown to be possible in a very general context by Dougherty and Foreman [1]; in particular, S^2 is (discretely) Baire paradoxical using six pieces with respect to the group SO_3 of rotations of $\mathbb{R}^3$ leaving the origin fixed. Thus, six is the optimal number of pieces required to realize a Baire-paradoxical decomposition of S^2.

Proof. Suppose, for example, that $m = 2$. Furthermore, without loss of generality, $g_1 = 1$. Thus $g_2\varphi_2 = 1_X - \varphi_1 = \sum_{i=2}^{m+n} \varphi_i \geq \varphi_2$ (B-a.e.), whence $g_2\varphi_2 = \varphi_2$ (B-a.e.) by the lemma. Thus $\varphi_i = 0$ (B-a.e.) for all $i \geq 3$, which contradicts the fact that X is not of first category in itself. □

Note that the same argument generalizes (with the same proof) to the case where X is a metric space which is *precompact* (i.e., for each $r > 0$ it can be covered by finitely many balls of radius r)—so that convergent sequences are replaced by Cauchy sequences in the proof of the lemma—and which is not of first category in itself.

This result has to be put in contrast with the fact that S^2 is paradoxical using four, and not fewer than four, pieces (which do not necessarily satisfy the Baire property) (see [3]). Thus it is (much!) more difficult to realize paradoxes with pieces satisfying the property of Baire (even, as in this paper, modulo meager sets) than paradoxes with unrestricted pieces.

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