AN EXAMPLE CONCERNING
THE YOSIDA-HEWITT DECOMPOSITION
OF FINITELY ADDITIVE MEASURES

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Abstract. Let \( \lambda \) be Lebesgue measure on the Lebesgue \( \sigma \)-algebra \( \mathcal{L} \) of
\( I := [0, 1] \). The author gives an example of a purely finitely additive mea-
sure \( \varphi : \mathcal{L} \to [0, 1] \) vanishing on \( \lambda \)-null sets such that \( \int f d\varphi = \int f d\lambda \)
for every bounded continuous function \( f \) on \( I \) (\( f \in C_b(I) \)). Consequently,
\( \lambda - \varphi \in L^\infty(\lambda)' \) annihilates \( C_b(I) \) and is not purely finitely additive, contrary
to an assertion of Yosida and Hewitt.

To be in accordance with [HY], the scalar field is \( \mathbb{R} \), although \( \mathbb{C} \) could be
used with insignificant changes in what follows. Also, only bounded real-valued
set functions are considered. A finitely additive measure \( \varphi \geq 0 \) on an algebra
of sets is called purely finitely additive (p.f.a.) if every \( \sigma \)-additive measure \( \mu \),
\( 0 \leq \mu \leq \varphi \), is zero. If the requirement \( \varphi \geq 0 \) is dropped, \( \varphi \) is called p.f.a. if
the positive and negative variations \( \varphi_+ \) and \( \varphi_- \) are both p.f.a. Every finitely
additive measure can be uniquely written as \( \mu + \varphi \), where \( \mu \) is \( \sigma \)-additive and
\( \varphi \) is p.f.a. This is the content of the Yosida-Hewitt decomposition theorem
[HY, 1.24]. In later sections of this fundamental work on finitely additive
measures, the authors study the bounded linear functionals on \( L^\infty(\lambda) \). Those
are easily identified as the finitely additive measures on \( \mathcal{L} \) vanishing on \( \lambda \)-null
sets [HY, 2.3]. Such a measure \( \varphi \) is p.f.a. if and only if it is concentrated on
sets of arbitrarily small \( \lambda \)-measure, where \( \varphi \) is called concentrated on \( E \in \mathcal{L} \)
if \( \|\varphi\|(I\setminus E) = 0 \) [HY, 3.1].

Let us turn to the construction of the example announced in the abstract.
For any \( t_0 \in I \) there exists a positive (hence bounded) linear functional \( \varphi_0 \) on
\( L^\infty(\lambda) \) with \( \varphi_0|C_b(I) = \delta_{t_0} := \) point evaluation at \( t_0 \). Just apply the Hahn-
Banach theorem to find a linear functional below the sublinear functional \( f \mapsto \text{ess}\limsup_{t \to t_0} f(t) := \lim_{t \downarrow t_0} \text{ess}\sup_{t_0 - \epsilon \leq t \leq t_0 + \epsilon} f(t) \) on \( L^\infty(\lambda) \). [One could even
find a character of \( L^\infty(\lambda) \) extending \( \delta_{t_0} \) (see, e.g., [IT, p. 107], but I will not
need this.) Any such \( \varphi_0 \) is concentrated on every neighbourhood \( U \) of \( t_0 \) (in
particular p.f.a.). To see this, take any \( f \in C_b(I) \), \( f \leq 1_U \), \( f(t_0) = 1 \). Then
\( 1 = \delta_{t_0}(f) = \varphi_0(f) \leq \varphi_0(1_U) \leq \varphi_0(1) = 1 \), so \( \varphi_0(1_U) = 1 \).
Now let \( t_{nj} := (j + \frac{1}{2})2^{-n} \) for \( n \in \mathbb{N}, 0 \leq j \leq 2^n - 1 \), and \( \varphi_{nj} \) be a positive functional on \( L^\infty(\lambda) \) with \( \varphi_{nj}|_{C_b(I)} = \delta_{t_{nj}} \). Next, choose and fix a Banach limit \( \text{Lim} \), that is a positive (hence bounded) linear extension of the “lim” functional over \( l^\infty \). (Take a linear functional below \( \text{lim sup} \).) For \( x = (x_n)_{n \in \mathbb{N}} \in l^\infty \) write \( \text{Lim}_{n \to \infty} x_n \) instead of \( \text{Lim}(x) \).

Define \( \varphi(f) := \text{Lim}_{n \to \infty} 2^{-n} \sum_{j=0}^{2^n-1} \varphi_{nj}(f) \) for \( f \in L^\infty(\lambda) \). Obviously, \( \varphi \in L^\infty(\lambda)' \) is positive and concentrated on every neighbourhood of the set of points \( t_{nj} \); hence, \( \varphi \) is p.f.a. Moreover, any \( f \in C_b(I) \) arbitrarily extended over \([0, 1]\) is Riemann integrable over \([0, 1]\), so \( \varphi(f) = \int_0^1 f(t) \, dt = \int f \, d\lambda \). \( \varphi(1) \) being 1, the set function \( \varphi(A) = \varphi(1_A) \), \( A \in \mathcal{L} \), has all the announced properties.

Finally, the difference \( \lambda - \varphi \in L^\infty(\lambda)' \) annihilates \( C_b(I) \) but is not p.f.a. (uniqueness of the decomposition!), contrary to the last assertion of [HY, Theorem 3.4].

The careful reader might argue that Yosida and Hewitt consider \( \mathbb{R} \) instead of \( I \). However, note that their measure \( \lambda \) is an (arbitrary) probability measure equivalent to Lebesgue measure on \( \mathbb{R} \). Therefore, any both-way null set-preserving homeomorphism \( I \to \mathbb{R} \) can be used for translation.

**Noted added in proof**

After this note had been accepted for publication, I found that M. Valadier (Une singulière forme linéaire singulière sur \( L^\infty \), Sém. d’Analyse Convexe Montpellier 1987, Exposé no. 4) has (also) constructed a p.f.a. functional on \( L^\infty(\lambda) \) which extends the functional \( \lambda \) on \( C[0, 1] \). In this connection, see also the article by Y. A. Abramovich and A. W. Wickstead, *Singular extensions and restrictions of order continuous functionals*, Hokkaida Math. J. 21 (1992), 475–482.

**References**


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