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HOMOGENEOUS HAMILTONIAN G-BUNDLES

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ABSTRACT. The Kirillov, Kostant, Souriau, et al. theorem on homogeneous Hamiltonian G-manifolds is generalized to (closed) Hamiltonian G-manifolds with isotropy groups of (fixed rank) maximal rank.

Let (P, ω) be a symplectic manifold, and let G be a compact Lie group of symplectomorphisms. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. We suppose the action has a momentum map $J: P \to \mathfrak{g}^*$; that is, J is a smooth map such that if we define $\hat{J}: \mathfrak{g} \to C^\infty(P)$ by $\hat{J}(A)(x) = J(x)(A)$, $A \in \mathfrak{g}$, then $d\hat{J}(A) = i(A_\#)\omega$, $A_\#$ the vector field on P generated by the action of $\exp tA$ on P. We also assume that J is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$. (P, ω, J) is then called a *Hamiltonian G-manifold.* Without loss of generality, we can assume P/G is connected.

The isotropy subgroups of the coadjoint action of a compact Lie group G on $\mathfrak{g}^*$ contain a maximal torus, and hence have maximal rank. The well-known result [1] of Kirillov, Kostant, Souriau, et al. that for any homogeneous Hamiltonian G-space P the momentum map $J: P \to \mathfrak{g}^*$ is a covering map onto $G\mu$ for some $\mu \in \mathfrak{g}^*$, implies all isotropy subgroups have maximum rank. We also note the trivial fact that a Hamiltonian G-manifold with G finite has maximum rank isotropy subgroups.

1. PRINCIPAL ORBIT TYPE

If M is a (locally) smooth G-manifold such that $M_* = M/G$ is connected and G is compact, there is [3] an isotropy subgroup H such that every isotropy subgroup contains a conjugate of H. The conjugacy class (H) is called the *principal orbit type* of M.

For any isotropy subgroup K of G, let $M_{(K)} = \{x \in M | (G_X) = (K)\}$. Then $M_{(K)}$ is a G-invariant submanifold. Let $M_K = \{x \in M | G_X = K\}$; i.e., $M_K = M_{(K)} \cap M^K$. M^K and M_K are $N(K)$-invariant submanifolds, $N(K)$ the normalizer of K in G.

Remark. Since K acts trivially on M^k, the $N(K)$-manifolds M^K and M_K are $(\bar{N}(K) = N(K)/K)$-manifolds. Also, M_K is a free $\bar{N}(K)$-manifold.

We refer to [4] for the next two results.

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Lemma 1.0. *Let (H) be the principal orbit type of M. The closure $\bar{M}_H$ of M_H is the union of those components of M^H which meet $M_{(H)}$. The components of $\bar{M}_H$ all have the same dimension, and $\bar{M}_H$ is an $\bar{N}(H)$-submanifold. $\bar{M}_H$ is called the principal part of M.*

Principal Orbit Theorem 1.1. *Let (H) be the principal orbit type of M. The inclusion $i\colon \bar{M}_H \to M$ induces a homeomorphism $i_*\colon \bar{M}_H/N(H) \to M/G$. In particular, $M = G(\bar{M}_H)$.*

2. The momentum map and orbit types

Lemma 2.0 [5]. *If G acts as symplectomorphisms on (P, ω), then we may give P an invariant Riemannian metric μ and an almost complex structure j such that $\omega(X, Y) = \mu(jX, Y)$.*

Corollary 2.1. *j is G-invariant; i.e., $g_*^{-1} j g_* = j$, $g \in G$. Thus P^G is an almost complex submanifold and hence a symplectic submanifold; i.e., $\omega|P^G$ is nondegenerate.*

Corollary 2.1 implies

Proposition 2.2. *Let (P, ω, J) be a Hamiltonian G-manifold. Let K be any isotropy subgroup of G on P. $(P^K, \omega|P^K)$ is an $\bar{N}(K)$ symplectic manifold. Since J is equivariant, J maps P^K to $\mathfrak{g}^{*K}$, and $J^K = J|P^K$ is $\bar{N}(K)$ invariant.*

Give $\mathfrak{g}$ an $\mathrm{Ad}(G)$ invariant metric. Let $\mathfrak{z}(K)$ be the Lie algebra of the centralizer of K in G. Then $\mathfrak{g} \cong \mathfrak{z}(K) \oplus \mathfrak{z}(K)^\perp$ as $\mathrm{Ad}(N(K))$ representations.

Lemma 2.3. $\mathrm{Im}\, J^K \subset \mathfrak{z}(K)^*$; *i.e., $J^K(x)$, $x \in P^K$, vanishes on $\mathfrak{z}(K)^\perp$. Furthermore, $J^K\colon P^K \to \mathfrak{z}(K)^*$ is $\bar{N}(K)$-invariant.*

Proof. If $x \in P^K$, $k \in K$, and $A \in \mathfrak{g}$, then

$$J^K(x)(A) = J^K(kx)(A) = (\mathrm{Ad}^*(k^{-1})J^K(x))(A) = J^K(x)(\mathrm{Ad}(k)A).$$

If $A \in \mathfrak{z}(K)^\perp$ and $A \neq 0$, then $\mathrm{Ad}(k)(A) \neq A$ for some $k \in K$, and $J^K(x)(A - \mathrm{Ad}(k)(A)) = 0$. But $\ker J^K(x)$ is an $\mathrm{Ad}(K)$-invariant subspace of $\mathfrak{g}$. Since $\ker J^K(x)$ is nontrivial on each such subspace of $\mathfrak{z}(K)^\perp$, it must contain any irreducible subspace of $\mathfrak{z}(K)^\perp$ and hence $\mathfrak{z}(K)^\perp$.

Let $\mathfrak{c}(K)$ be the Lie algebra of the center of K.

Lemma 2.4. *There is a $\lambda \in \mathfrak{c}(K)^*$ such that, for all $x \in P^K$, $J^K(x)|\mathfrak{c}(K) = \lambda$.*

Proof. Let $A \in \mathfrak{c}(K)$. Then $d\hat{J}(A) = 0$ on P^K since the associated vector field $A_\#$ is trivial on P^K. Hence $\hat{J}(A)$ is constant on P^K; i.e., $J^K(x)(A) = \lambda(A)$, $A \in \mathfrak{c}(K)$, for some fixed $\lambda \in \mathfrak{c}(K)^*$.

3. Homogeneous Hamiltonian G-bundles

A Hamiltonian G-manifold (P, ω, J) is called a *Hamiltonian G-bundle* if P is a smooth equivariant bundle $p : P \to B$ over a G-manifold B and J factors through B; i.e., $J = J_0 \circ p$, $J_0\colon B \to \mathfrak{g}^*$. (Note that this implies that J_0 is equivariant.) A Hamiltonian G-bundle is called *homogeneous* if B is homogeneous.

Theorem 3.0. *Let (P, ω, J) be a Hamiltonian G-manifold, G compact. Then the following are equivalent.*

(a) *All isotropy subgroups of P have maximum rank.*

(b) *If (H) is the principal orbit type of P, $N(H)/H$ is finite.*

(c) *The image of J is a single coadjoint orbit.*

(d) *(P, ω, J) is a homogeneous Hamiltonian G-bundle.*

(e) *$P \simeq \bar{G} \times_{\bar{K}} Q$, where $K = G_\sigma$, $\bar{K} = K/K_0$, $\bar{G} = G/K_0$, with $J: P \to \bar{G}/\bar{K} = G/K = G\sigma$ the projection and $Q = J^{-1}(\sigma)$. Furthermore, if $\Pi: \bar{G} \times Q \to \bar{G} \times_{\bar{K}} Q$ is the projection, then $\Pi^*\omega = (\tilde{\nu}, \omega_0)$, where $\omega_0 = \omega|Q$ is a $\bar{K}$ symplectic manifold and $\tilde{\nu} = q^*\nu$, with $q: \bar{G} \to \bar{G}/\bar{K} = G\sigma$ the quotient map and ν the Kirillov symplectic structure on $G\sigma$.*

Proof. (a) $\Rightarrow$ (b) If all isotropy subgroups have maximal rank, H has maximal rank and $N(H)/H$ is finite.

(b) $\Rightarrow$ (c) $\mathfrak{z}(H)/\mathfrak{c}(H) = \bar{n}(H) = 0$. By (2.4), $J^H(P^H)|\mathfrak{c}(H)) \equiv \lambda$. Extend λ to $\tilde{\lambda}$ on $\mathfrak{g}^*$ by $\tilde{\lambda} = 0$ on $\mathfrak{z}(H)^\perp$. Since $P = G(P^H)$ and J is equivariant, $J(P) = G\tilde{\lambda}$.

(c) $\Leftrightarrow$ (d) Since the pullback of a generalized slice by an equivariant map is a generalized slice, $P = J^{-1}(G\sigma) = G \times_K Q$, $Q = J^{-1}(\sigma)$, $K = G_\sigma$. Thus $p: P \to G/K$ is an equivariant bundle and $J = J_0 \circ p$, with $J_0: G/K = G\sigma \to \mathfrak{g}^*$ the inclusion. The converse is trivial.

(d) $\Rightarrow$ (a) Consider $J|Gx$, $x \in J^{-1}(\sigma)$, where $J(P) = G\sigma$. If, for some $A \in \mathfrak{g}$, $A_\#(x) \in \operatorname{Ker}\{d_x J: T_x(Gx) \to T_\sigma(G\sigma)\}$, then $\mu_x(jA_\#, B_\#) = \omega_x(A_\#, B_\#) = d_x J(A_\#)(B) = 0$ for all $B \in \mathfrak{g}$. Now writing $K = G_\sigma$, we have $J^{-1}(G\sigma) = G \times_K J^{-1}(\sigma)$ and $J^{-1}(\sigma) \supset K \times_L V$, with $L = G_x$ and V a transverse disc to Kx and hence Gx at x. Since $jA_\#(x)$ is orthogonal to $T_x(Gx)$ and V is transverse to $T_x(GX)$, there is a $B \in \mathfrak{g}$ such that $jA_\#(x) + B_\#(x) \in T_x V$ and hence $d_x J(jA_\# + B_\#) = 0$. Then

$$\mu_x(A_\#, A_\#) = \mu_x(A_\# - jB_\#, A_\#) = -d_x(jA_\# + B_\#)(A) = 0.$$

Hence $A_\#(x) = 0$ and $J|Gx$ is a covering map. But then G_x has the same rank as G_σ and hence is of maximum rank.

Remark. The above proof that J is a covering map on each orbit of a homogeneous Hamiltonian G-bundle does not require that G is compact.

(d) $\Leftarrow$ (e) Since (d) $\Rightarrow$ (c), $J(P) = G\sigma$. Hence $P = J^{-1}(G\sigma) \simeq G \times_K Q$. Since $Q/K = P/G$ is assumed connected, Q has a principal orbit type, which must be that of P. Since (d) implies J is a covering map on each orbit (see proof (d) $\Rightarrow$ (a)), K/H is finite and $K_0 = H_0$. Since $Q = K\bar{Q}_H = \bar{K}\bar{Q}_H$, $\bar{K}$ is finite, and all components of $\bar{Q}_H$ have the same dimension, we claim $Q = \bar{K} \times_L \bar{Q}_H$, where L is the subgroup of K leaving $\bar{Q}_H$ invariant. In fact, $\dim \bar{Q}_H = \dim Q$ and $\bar{Q}_H$ is both open and closed in Q, and hence the disjoint union of components of Q. Now $kQ_H = Q_{kHk^{-1}}$, $k \in K$, and $Q_{kHk^{-1}}$ and $Q_{k'Hk'^{-1}}$ meet iff $kHk^{-1} = k'Hk'^{-1}$. Since Q_H is open and dense in $\bar{Q}_H$ and hence in the components of Q contained in $\bar{Q}_H$, $Q_{kHk^{-1}}$ and $Q_{k'Hk'^{-1}}$ meet the same component of Q iff they intersect. Thus $\bar{Q}_H$ meets $k\bar{Q}_H = \bar{Q}_{kHk^{-1}}$ iff $k \in N(H)$. This proves the assertion and shows that $L = N(H) \cap K$.

Now $P = G \times_K (K \times_L \bar{Q}_H) = G \times_L \bar{Q}_H$ and $P_{(H)} = G \times_L Q_H$. Hence $P_H = N(H) \times_L Q_H$, and $\bar{P}_H = N(H) \times_L \bar{Q}_H = \bar{N}(H) \times_L \bar{Q}_H$. Since $(\bar{P}_H, \omega|\bar{P}_H)$ is

an $\bar{N}(H)$ symplectic manifold and $\bar{Q}_H$ is an open submanifold, $(\bar{Q}_H, \omega|\bar{Q}_H)$ is an $\bar{L}$ symplectic manifold. Hence $(Q, \omega|Q)$ is a $\bar{K}$ symplectic manifold.

Let $\tilde{\omega} = \Pi^*\omega$, with $\Pi : \bar{G} \times Q \to \bar{G} \times_{\bar{K}} Q$ the projection. Let $(\bar{g}, x) \in \bar{G} \times Q$ and $A, B \in \mathfrak{g}$. Then

$$\tilde{\omega}(A_\#, B_\#)(\bar{g}, x) = \omega(A_\#, B_\#)([g, x]) = d_{[g,x]}J(A_\#)(B) = \nu(A_\#, B_\#)(\bar{g}).$$

For X, Y tangent to $\bar{g} \times Q$,

$$\tilde{\omega}(X, Y)(\bar{g}, x) = \omega(g_*X, g_*Y)([g, x]) = \omega(X, Y)(x) = (\omega|Q)(X, Y)(x)$$

by G-invariance. Hence, $\tilde{\omega} = (\tilde{\nu}, \omega|Q)$.

Conversely, (e) $\Rightarrow$ (d) by definition.

Remark. If G is connected, $K = G_\sigma$ is connected [3, Chapter I] and $\bar{K} = 1$, so $P = \bar{G} \times Q$. Also, $H = L = K$, so $Q = \bar{Q}_H$ and $\bar{P}_H = \bar{N}(H) \times \bar{Q}_H$.

Addendum to Theorem 3.0. *If P is compact,* (a) *may be replaced by*:

(a′) *All isotropy subgroups of P have the same rank.*

Proof. If all isotropy subgroups have the same rank and $x \in \bar{P}_H$, then $\bar{N}(H)_x = (G_x \cap N(H))/H$ is a compact Lie group of rank zero; i.e., a finite group. Thus $(\bar{P}_H, \omega, \bar{J}_H)$ is a compact Hamiltonian $\bar{N}(H)$-manifold with finite isotropy subgroups. As proved in Lemma 3.1 this implies $\bar{N}(H)$ is finite, and hence the equivalence of (a′) with (a) through (e).

Lemma 3.1. *Let (P, ω, J) be a compact Hamiltonian G-manifold with finite isotropy subgroups. Then G is finite.*

Proof. J is a submersion since $d_xJ(jA_\#(x))(A) = \omega_x(jA_\#, A_\#) = -\mu_x(A_\#, A_\#) \neq 0$ and $\dim j(T_xGx) = \dim \mathfrak{g}^*$. Thus $J(P)$ is open. Since P is compact, $J(P)$ is also closed, and hence $J(P) = \mathfrak{g}^*$. But this implies $\mathfrak{g}^*$ is compact, and hence $\mathfrak{g}^* = (0)$; i.e., G is finite.

The result below shows that Theorem 3.0 completely characterizes homogeneous Hamiltonian G-bundles.

Theorem 3.2. *Let K be an isotropy subgroup of* $\mathrm{Ad}(G)$ *on* $\mathfrak{g}^*$, *and let* (Q, ω_0) *be a* $(\bar{K} = K/K_0)$*-symplectic manifold. Then* $P = G \times_K Q = \bar{G} \times_{\bar{K}} Q$ *has a symplectic structure* ω *such that the projection* $J: P \to G/K$ *is a G-invariant momentum map.*

Proof. The projection map $\Pi: \bar{G} \times Q \to G \times_K Q$ is a covering map. Let ν be the Kirillov symplectic structure on G/K, and let $\tilde{\omega} = (q^*\nu, \omega_0)$, $q : \bar{G} \to G/K$. Then $(\bar{G} \times Q, \tilde{\omega})$ is a $(G \times \bar{K})$-symplectic manifold. Hence ω pushes down to a G-symplectic structure ω on P. Let $J: \bar{G} \times_{\bar{K}} Q \to G/K$ be the projection. Then if $A \in \mathfrak{g}$, $X \in T_xP$, and $\Pi_*(\tilde{X}) = X$,

$$(d_xJ)(X)(A) = \nu(J_*(X), J_*(A_\#)) = \tilde{\omega}(\tilde{X}, A_\#) = \omega(X, A_\#),$$

since if $\tilde{X} = (\tilde{X}_1, \tilde{X}_2) \in T(\bar{G} \times Q)$, then $A_\# = (A_\#, 0)$ and $\omega_0(\tilde{X}_2, 0) = 0$. Hence J is a momentum map for (P, ω) and is evidently equivariant.

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