HOMOGENEOUS HAMILTONIAN $G$-BUNDLES

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Abstract. The Kirillov, Kostant, Souriau, et al. theorem on homogeneous Hamiltonian $G$-manifolds is generalized to (closed) Hamiltonian $G$-manifolds with isotropy groups of (fixed rank) maximal rank.

Let $(P, \omega)$ be a symplectic manifold, and let $G$ be a compact Lie group of symplectomorphisms. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. We suppose the action has a momentum map $J: P \to \mathfrak{g}^*$; that is, $J$ is a smooth map such that if we define $\hat{J}: \mathfrak{g} \to C^\infty(P)$ by $\hat{J}(A)(x) = J(x)(A)$, $A \in \mathfrak{g}$, then $d\hat{J}(A) = i(A_#)\omega$, $A_#$ the vector field on $P$ generated by the action of $\exp tA$ on $P$. We also assume that $J$ is equivariant with respect to the coadjoint action of $G$ on $\mathfrak{g}^*$. $(P, \omega, J)$ is then called a Hamiltonian $G$-manifold. Without loss of generality, we can assume $P/G$ is connected.

The isotropy subgroups of the coadjoint action of a compact Lie group $G$ on $\mathfrak{g}^*$ contain a maximal torus, and hence have maximal rank. The well-known result [1] of Kirillov, Kostant, Souriau, et al. that for any homogeneous Hamiltonian $G$-space $P$ the momentum map $J: P \to \mathfrak{g}^*$ is a covering map onto $G\mu$ for some $\mu \in \mathfrak{g}^*$, implies all isotropy subgroups have maximum rank. We also note the trivial fact that a Hamiltonian $G$-manifold with $G$ finite has maximum rank isotropy subgroups.

1. Principal orbit type

If $M$ is a (locally) smooth $G$-manifold such that $M_* = M/G$ is connected and $G$ is compact, there is [3] an isotropy subgroup $H$ such that every isotropy subgroup contains a conjugate of $H$. The conjugacy class $(H)$ is called the principal orbit type of $M$.

For any isotropy subgroup $K$ of $G$, let $M_{(K)} = \{x \in M | (Gx) = (K)\}$. Then $M_{(K)}$ is a $G$-invariant submanifold. Let $M_K = \{x \in M | Gx = K\}$; i.e., $M_K = M_{(K)} \cap M^K$. $M^K$ and $M_K$ are $N(K)$-invariant submanifolds, $N(K)$ the normalizer of $K$ in $G$.

Remark. Since $K$ acts trivially on $M^K$, the $N(K)$-manifolds $M^K$ and $M_K$ are $(\tilde{N}(K) = N(K)/K)$-manifolds. Also, $M_K$ is a free $\tilde{N}(K)$-manifold.

We refer to [4] for the next two results.

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Lemma 1.0. Let \((H)\) be the principal orbit type of \(M\). The closure \(\bar{M}_H\) of \(M_H\) is the union of those components of \(M^H\) which meet \(M(H)\). The components of \(\bar{M}_H\) all have the same dimension, and \(\bar{M}_H\) is an \(N(H)\)-submanifold. \(\bar{M}_H\) is called the principal part of \(M\).

Principal Orbit Theorem 1.1. Let \((H)\) be the principal orbit type of \(M\). The inclusion \(i: \bar{M}_H \to M\) induces a homeomorphism \(i_*: \bar{M}_H/N(H) \to M/G\). In particular, \(M = G(\bar{M}_H)\).

2. The momentum map and orbit types

Lemma 2.0 [5]. If \(G\) acts as symplectomorphisms on \((P, \omega)\), then we may give \(P\) an invariant Riemannian metric \(\mu\) and an almost complex structure \(j\) such that \(\omega(X, Y) = \mu(jX, Y)\).

Corollary 2.1. \(j\) is \(G\)-invariant; i.e., \(g^{-1}jg = j\), \(g \in G\). Thus \(P^G\) is an almost complex submanifold and hence a symplectic submanifold; i.e., \(\omega|P^G\) is nondegenerate.

Corollary 2.1 implies

Proposition 2.2. Let \((P, \omega, J)\) be a Hamiltonian \(G\)-manifold. Let \(K\) be any isotropy subgroup of \(G\) on \(P\). \((P^K, \omega|P^K)\) is an \(N(K)\) symplectic manifold. Since \(J\) is equivariant, \(M\) maps \(P^K\) to \(g^*\), and \(J^K = J|P^K\) is \(N(K)\) invariant.

Give \(g\) an \(Ad(G)\) invariant metric. Let \(\mathfrak{z}(K)\) be the Lie algebra of the centralizer of \(K\) in \(G\). Then \(\mathfrak{g} \cong \mathfrak{z}(K) \oplus \mathfrak{z}(K)^\perp\) as \(Ad(N(K))\) representations.

Lemma 2.3. \(\operatorname{Im} J^K \subset \mathfrak{z}(K)^\ast\); i.e., \(J^K(x), x \in P^K\), vanishes on \(\mathfrak{z}(K)^\perp\). Furthermore, \(J^K: P^K \to \mathfrak{z}(K)^\ast\) is \(N(K)\)-invariant.

Proof. If \(x \in P^K\), \(k \in K\), and \(A \in \mathfrak{g}\), then

\[
\]

If \(A \in \mathfrak{z}(K)^\perp\) and \(A \neq 0\), then \(Ad(k)(A) \neq A\) for some \(k \in K\), and \(J^K(x)(A - Ad(k)(A)) = 0\). But \(\ker J^K(x)\) is an \(Ad(K)\)-invariant subspace of \(\mathfrak{g}\). Since \(\ker J^K(x)\) is nontrivial on each such subspace of \(\mathfrak{z}(K)^\perp\), it must contain any irreducible subspace of \(\mathfrak{z}(K)^\perp\) and hence \(\mathfrak{z}(K)^\perp\).

Let \(c(K)\) be the Lie algebra of the center of \(K\).

Lemma 2.4. There is a \(\lambda \in c(K)^\ast\) such that, for all \(x \in P^K\), \(J^K(x)|c(K) = \lambda\).

Proof. Let \(A \in c(K)\). Then \(dJ(A) = 0\) on \(P^K\) since the associated vector field \(A_\#\) is trivial on \(P^K\). Hence \(J(A)\) is constant on \(P^K\); i.e., \(J^K(x)(A) = \lambda(A), A \in c(K)\), for some fixed \(\lambda \in c(K)^\ast\).

3. Homogeneous Hamiltonian \(G\)-bundles

A Hamiltonian \(G\)-manifold \((P, \omega, J)\) is called a Hamiltonian \(G\)-bundle if \(P\) is a smooth equivariant bundle \(p: P \to B\) over a \(G\)-manifold \(B\) and \(J\) factors through \(B\); i.e., \(J = J_0 \circ p\), \(J_0: B \to g^\ast\). (Note that this implies that \(J_0\) is equivariant.) A Hamiltonian \(G\)-bundle is called homogeneous if \(B\) is homogeneous.
Theorem 3.0. Let \((P, \omega, J)\) be a Hamiltonian \(G\)-manifold, \(G\) compact. Then the following are equivalent.

(a) All isotropy subgroups of \(P\) have maximum rank.
(b) If \((H)\) is the principal orbit type of \(P\), \(N(H)/H\) is finite.
(c) The image of \(J\) is a single coadjoint orbit.
(d) \((P, \omega, J)\) is a homogeneous Hamiltonian \(G\)-bundle.
(e) \(P \simeq G \times_K Q\), where \(K = G_\sigma\), \(K = K/K_0\), \(G = G/K_0\), with \(J: P \to G/K = G/K = G\sigma\) the projection and \(Q = J^{-1}(\sigma)\). Furthermore, if \(\Pi: G \times Q \to G \times_K Q\) is the projection, then \(\Pi^* \omega = (\tilde{\nu}, \omega_0)\), where \(\omega_0 = \omega|Q\) is a \(K\) symplectic manifold and \(\tilde{\nu} = q^*\nu\), with \(q: G \to G/K = G\sigma\) the quotient map and \(\nu\) the Kirillov symplectic structure on \(G\sigma\).

Proof. (a) \(\Rightarrow\) (b) If all isotropy subgroups have maximal rank, \(H\) has maximal rank and \(N(H)/H\) is finite.

(b) \(\Rightarrow\) (c) \(3H)/c(H) = n(H) = 0\). By (2.4), \(J^H(P^H)c(H) = \lambda\). Extend \(\lambda\) to \(\tilde{\lambda}\) on \(g^*\) by \(\tilde{\lambda} = 0\) on \(3H)^\bot\). Since \(P = G(P^H)\) and \(J\) is equivariant, \(J(P) = G\tilde{\lambda}\).

(c) \(\Leftrightarrow\) (d) Since the pullback of a generalized slice by an equivariant map is a generalized slice, \(P \simeq J^{-1}(\sigma) = G \times_K Q\), \(Q = J^{-1}(\sigma)\), \(K = G_\sigma\). Thus \(P: P \to G/K\) is an equivariant bundle and \(J = J_0 \circ p\), with \(J_0: G/K = G\sigma \to g^*\), the inclusion. The converse is trivial.

(d) \(\Rightarrow\) (a) Consider \(J|Gx\), \(x \in J^{-1}(\sigma)\), where \(J(P) = G\sigma\). If, for some \(A \in g\), \(A_A(x) \in \ker\{d_Ax: T_x(Gx) \to T_\sigma(G\sigma)\}\), then \(\mu_x(jA_\# B_\#) = \omega_x(A_\#, B_\#) = d_xJ(A_\#)(B) = 0\) for all \(B \in g\). Now writing \(K = G_\sigma\), we have \(J^{-1}(\sigma) = G \times_K J^{-1}(\sigma)\) and \(J^{-1}(\sigma) \supset K \times_L V\), with \(L = G_x\) and \(V\) a transverse disc to \(Kx\) and hence \(Gx\) at \(x\). Since \(jA_\#(x)\) is orthogonal to \(T_x(Gx)\) and \(V\) is transverse to \(T_x(Gx)\), there is a \(B \in g\) such that \(jA_\#(x) + B_\#(x) \in T_xV\) and hence \(d_xJ(jA_\# + B_\#) = 0\). Then

\[\mu_x(A_\#, A_\#) = \mu_x(A_\# - jB_\#, A_\#) = -d_x(jA_\# + B_\#)(A) = 0.\]

Hence \(A_\#(x) = 0\) and \(J|Gx\) is a covering map. But then \(G_x\) has the same rank as \(G\sigma\) and hence is of maximum rank.

Remark. The above proof that \(J\) is a covering map on each orbit of a homogeneous Hamiltonian \(G\)-bundle does not require that \(G\) is compact.

(d) \(\Leftarrow\) (e) Since (d) \(\Rightarrow\) (c), \(J(P) = G\sigma\). Hence \(P \simeq J^{-1}(\sigma) \simeq G \times_K Q\). Since \(Q/K = P/G\) is assumed connected, \(Q\) has a principal orbit type, which must be that of \(P\). Since (d) implies \(J\) is a covering map on each orbit (see proof (d) \(\Rightarrow\) (a)), \(K/H\) is finite and \(K_0 = H_0\). Since \(Q = KQ_H = KQ_H\), \(K\) is finite, and all components of \(Q_H\) have the same dimension, we claim \(Q = K \times_L \tilde{Q}_H\), where \(L\) is the subgroup of \(K\) leaving \(\tilde{Q}_H\) invariant. In fact, \(\dim \tilde{Q}_H = \dim Q\) and \(\tilde{Q}_H\) is both open and closed in \(Q\), and hence the disjoint union of components of \(Q\). Now \(kQ_H = Q_{kH^{-1}}\), \(k \in K\), and \(Q_{kH^{-1}}\) and \(Q_{k' H'^{-1}}\) meet iff \(kHk^{-1} = k'H^{-1}\). Since \(Q_H\) is open and dense in \(\tilde{Q}_H\) and hence in the components of \(Q\) contained in \(\tilde{Q}_H\), \(Q_{kH^{-1}}\) and \(Q_{k' H'^{-1}}\) meet the same component of \(Q\) iff they intersect. Thus \(\tilde{Q}_H\) meets \(k\tilde{Q}_H = Q_{kH^{-1}}\) iff \(k \in N(H)\). This proves the assertion and shows that \(L = N(H) \cap K\).

Now \(P = G \times_K (K \times_L \tilde{Q}_H) = G \times_L \tilde{Q}_H\) and \(P_H = G \times_L Q_H\). Hence \(P_H = N(H) \times_L Q_H\), and \(P_H = N(H) \times_L \tilde{Q}_H = N(H) \times_L \tilde{Q}_H\). Since \((P_H, \omega|P_H)\) is...
an \( \hat{N}(H) \) symplectic manifold and \( \hat{Q}_H \) is an open submanifold, \( (Q_H, \omega|Q_H) \) is an \( L \) symplectic manifold. Hence \( Q, \omega|Q \) is a \( K \) symplectic manifold.

Let \( \tilde{\omega} = \Pi^*\omega \), with \( \Pi : \hat{G} \times Q \to \hat{G} \times_K Q \) the projection. Let \( (g, x) \in \hat{G} \times Q \) and \( A, B \in \mathfrak{g} \). Then

\[
\tilde{\omega}(A^g, B^g)(g, x) = \omega(A^g, B^g)([g, x]) = d_{[g, x]} J(A^g)(B) = \nu(A^g, B^g)(g).
\]

For \( X, Y \) tangent to \( \hat{g} \times Q \),

\[
\tilde{\omega}(X, Y)(g, x) = \omega(g \ast X, g \ast Y)([g, x]) = \omega(X, Y)(x) = (\omega|Q)(X, Y)(x)
\]

by \( G \)-invariance. Hence, \( \tilde{\omega} = (\nu, (\omega|Q)) \).

Conversely, \( (e) \Rightarrow (d) \) by definition.

**Remark.** If \( G \) is connected, \( K = G_0 \) is connected [3, Chapter I] and \( K = 1 \), so \( P = \hat{G} \times Q \). Also, \( H = L = K \), so \( Q = Q_H \) and \( P_H = \hat{N}(H) \times Q_H \).

**Addendum to Theorem 3.0.** If \( P \) is compact, \( (a) \) may be replaced by:

\( (a') \) All isotropy subgroups of \( P \) have the same rank.

**Proof.** If all isotropy subgroups have the same rank and \( x \in P_H \), then \( \hat{N}(H)_x = (G_x \cap N(H))/H \) is a compact Lie group of rank zero; i.e., a finite group. Thus \( (P_H, \omega, J_H) \) is a compact Hamiltonian \( \hat{N}(H) \)-manifold with finite isotropy subgroups. As proved in Lemma 3.1 this implies \( \hat{N}(H) \) is finite, and hence the equivalence of \( (a') \) with \( (a) \) through \( (e) \).

**Lemma 3.1.** Let \( (P, \omega, J) \) be a compact Hamiltonian \( G \)-manifold with finite isotropy subgroups. Then \( G \) is finite.

**Proof.** \( J \) is a submersion since \( d_x J(jA^g)(x)(A) = \omega_x jA^g, A^g) = -\mu_x(A^g, A^g) \neq 0 \) and \( \dim(T_xGx) = \dim g^* \). Thus \( J(P) \) is open. Since \( P \) is compact, \( J(P) \) is also closed, and hence \( J(P) = g^* \). But this implies \( g^* \) is compact, and hence \( g^* = \{0\} \); i.e., \( G \) is finite.

The result below shows that Theorem 3.0 completely characterizes homogeneous Hamiltonian \( G \)-bundles.

**Theorem 3.2.** Let \( K \) be an isotropy subgroup of \( \text{Ad}(G) \) on \( g^* \), and let \( (Q, \omega_0) \) be a \( (K = K/K_0) \)-symplectic manifold. Then \( P = G \times_K Q = G \times_K Q \) has a symplectic structure \( \omega \) such that the projection \( J : P \to G/K \) is a \( G \)-invariant momentum map.

**Proof.** The projection map \( \Pi : \hat{G} \times Q \to \hat{G} \times_K Q \) is a covering map. Let \( \nu \) be the Kirillov symplectic structure on \( G/K \), and let \( \tilde{\omega} = (q^*\nu, \omega_0) \), \( \tilde{\omega} : \hat{G} \to G/K \). Then \( (\hat{G} \times Q, \tilde{\omega}) \) is a \( (G \times K) \)-symplectic manifold. Hence \( \omega \) pushes down to a \( G \)-symplectic structure \( \omega \) on \( P \). Let \( J : G \times_K Q \to G/K \) be the projection. Then if \( A \in \mathfrak{g}, X \in T_x P \), and \( \Pi_*(\tilde{X}) = X \),

\[
(d_x J)(X)(A) = \nu(J_x(X), J_x(A^g)) = \tilde{\omega}(\tilde{X}, A^g) = 0,
\]

since if \( \tilde{X} = (\tilde{X}_1, \tilde{X}_2) \in T(G \times Q) \), then \( A^g = (A^g, 0) \) and \( \omega_0(\tilde{X}_2, 0) = 0 \). Hence \( J \) is a momentum map for \( (P, \omega) \) and is evidently equivariant.

**References**


