ON A COUNTING FORMULA OF DJOKOVIĆ FOR ELEMENTS OF FINITE ORDER IN COMPACT LIE GROUPS

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ABSTRACT. Given a compact connected simple Lie group $\mathfrak{G}$ and a positive integer N relatively prime to the order of the Weyl group we give a counting formula for the number of conjugacy classes of elements x of order N in $\mathfrak{G}$ with the property that the N -cyclotomic field when viewed as a Galois extension of the field of characters of x has Galois group containing a fixed chosen cyclic group $\mathcal{G}$. The case $\mathcal{G} = \{1\}$ recovers a formula, due to Djoković, which counts the number of conjugacy classes of elements of order dividing N in $\mathfrak{G}$.

0. INTRODUCTION

In this paper we derive polynomial formulas for the number of conjugacy classes of elements of order N coprime with the order of the Weyl group W of a compact connected simple Lie group whose character values generate an extension of $\mathbb{Q}$ contained in certain number fields.

This result generalizes that of [DP] ([PW]) where the case $N = p^K$ ($N = p$, respectively) was treated. It also generalizes Djoković [Djk1].

Let $\mathfrak{G}$ be a (real) compact connected simple Lie group. As in [DP] and [PW] we define a lattice Γ corresponding to a fixed maximal torus $\mathfrak{T}$ of $\mathfrak{G}$ via the following short exact sequence

$$(1) \quad 0 \rightarrow \Gamma \rightarrow \mathfrak{t} \xrightarrow{\exp 2\pi i(\cdot)} \mathfrak{T} \rightarrow 1$$

where $\mathfrak{t} = it'$, with t' the Lie algebra of $\mathfrak{T}$, and $i = \sqrt{-1}$.

The Weyl group W of $\mathfrak{G}$ acts on the lattice Γ and hence on $V = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma$. Given an element $w \in W$, the multiplicity of the d th root of unity $\zeta_d := e^{2\pi i/d}$ as an eigenvalue of w in V is denoted by

$$(2) \quad f_d(w).$$

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(If ζ_d is not an eigenvalue of w , $f_d(w)$ is defined to be 0.) Note that $d|m_i + 1$ for some $1 \leq i \leq l$, where $m_1, \dots, m_l$ are the *exponents* of $\mathfrak{G}$. In particular [Spg, Theorem 3.4, PW, Theorem 2], $1 \leq d \leq h$ where h is the *Coxeter number* of $\mathfrak{G}$.

We define a “universal” polynomial $G(t_1, \dots, t_h) \in \mathbb{Q}[t_1, \dots, t_h]$ (with h the Coxeter number of $\mathfrak{G}$) as follows:

$$(3) \quad G(t_1, \dots, t_h) := \frac{1}{|W|} \sum_{k_1, \dots, k_h \in \mathbb{N}} a_{k_1, \dots, k_h} t_1^{k_1} \cdots t_h^{k_h}$$

where $a_{k_1, \dots, k_h}$ is the number of elements of W that have ζ_i as an eigenvalue of multiplicity k_i , for all $1 \leq i \leq h$. Equivalently, G is given by the average sum

$$(4) \quad G(t_1, \dots, t_h) := \frac{1}{|W|} \sum_{w \in W} t_1^{f_1(w)} \cdots t_h^{f_h(w)}.$$

We consider elements x of finite order of $\mathfrak{G}$. If $x \in \mathfrak{G}$ has order N , then the character value $\text{tr}_V(\rho(x))$ of any finite dimensional (complex) representation (ρ, V) of $\mathfrak{G}$ is contained in the cyclotomic field $\mathbb{Q}(\zeta_N)$ where $\zeta_N = e^{2\pi i/N}$. The subfield of $\mathbb{Q}(\zeta_N)$ generated by all such character values is called the *field of characters* of x and is denoted by $A(x)$. As in [DP], we then define the *Galois group* of x by

$$(5) \quad \text{Gal}(x) := \text{Gal}(\mathbb{Q}(\zeta_N)/A(x))$$

which we will view as a subgroup of $(\mathbb{Z}/N\mathbb{Z})^*$ by means of the natural description of $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$.

We fix a positive integer N with prime factorization

$$(6) \quad N = p_1^{K_1} \cdots p_s^{K_s}$$

where $p_1, \dots, p_s$ are distinct primes *not* dividing the order of the Weyl group W of $\mathfrak{G}$. We fix a *cyclic* subgroup $\mathcal{C}$ of $(\mathbb{Z}/N\mathbb{Z})^*$ as well as a generator

$$(7) \quad a \in \mathcal{C}$$

and we set, for $1 \leq i \leq s$,

$$(8) \quad n_i = \text{ord}(a_i)$$

in $(\mathbb{Z}/p_i^{K_i}\mathbb{Z})^*$, where a_i is the image of a under the canonical projection $(\mathbb{Z}/N\mathbb{Z})^* \rightarrow (\mathbb{Z}/p_i^{K_i}\mathbb{Z})^*$.

The main result of this paper is the following theorem.

Theorem. *Notation as above. The number of conjugacy classes of elements of $\mathfrak{G}$ of order N with $(a) \subseteq \text{Gal}(x)$ is given by*

$$\sum_{\varepsilon=(\varepsilon_1, \dots, \varepsilon_s) \in (\mathbb{Z}/2\mathbb{Z})^s} (-1)^{|\varepsilon|} G(q_1(\varepsilon), \dots, q_h(\varepsilon))$$

where $q_j(\varepsilon) = \prod_{i: n_i=j} p_i^{(K_i-\varepsilon_i)}$ and $|\varepsilon| = \sum_{i=1}^s \varepsilon_i$.

1. COUNTING WEYL GROUP ORBITS

In this section we recall the basic approach adopted in [DP, Pzl1, PW].

The Weyl group W of $\mathfrak{G}$ acts on the lattice Γ and hence on the module $\Gamma/N\Gamma$, for any positive integer N . Observe that W also acts on the set of *reduced* elements of $\Gamma/N\Gamma$; i.e., those elements of order N .

Proposition 1. (a) *Let x be an element of $\mathfrak{G}$ of order N . Let $\text{Gal}(x)$ be as in (5). Then $|\text{Gal}(x)|$ divides $(\phi(N), |W|)$. In particular, $|\text{Gal}(x)|$ divides $\prod_{i=1}^s (p_i - 1)$. Moreover, for any $g \in \text{Gal}(x)$, the order $\text{ord}(g)$ of g divides the exponent $e(W)$ of W (i.e., the least common multiple of the orders of elements of W).*

(b) *Let $\mathcal{G}$ be a subgroup of $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ (with $|\mathcal{G}|$ dividing $\prod_{i=1}^s (p_i - 1)$) and $\{a(\alpha)\}_{\alpha \in I}$ be a family of generators of $\mathcal{G}$ under the canonical isomorphism $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \approx (\mathbb{Z}/N\mathbb{Z})^*$. The conjugacy classes of elements x of $\mathfrak{G}$ of order N such that $\mathcal{G} \subseteq \text{Gal}(x)$ are in bijection with W -orbits of reduced elements z of $\Gamma/N\Gamma$ such that, for any $\alpha \in I$, there exists some $w(\alpha) \in W$ for which $w(\alpha)z = a(\alpha)z$.*

Proof. From [DP, Proposition 1(b)], if x corresponds to $z \in \Gamma/N\Gamma$, then there is an isomorphism $W_z^e/W_z \xrightarrow{\sim} \text{Gal}(x)$ where $W_z \subseteq W_z^e$ are certain subgroups of W . Hence, $|\text{Gal}(x)|$ divides $|W|$. Of course $|\text{Gal}(x)|$ divides also $|(\mathbb{Z}/N\mathbb{Z})^*| = \phi(N) = \prod_{i=1}^s p_i^{K_i-1}(p_i - 1)$ (from (6)). Since $(N, |W|) = 1$ we conclude that $|\text{Gal}(x)|$ divides $\prod_{i=1}^s (p_i - 1)$. Finally, $e(W)$ kills any element of W , hence, of W_z^e/W_z , and hence, of $\text{Gal}(x)$. Thus $\text{ord}(g) | e(W)$. This proves part (a).

[Pzl1] shows the existence of a 1-1 correspondence between conjugacy classes of elements of $\mathfrak{G}$ of order N and W -orbits of reduced elements of $\Gamma/N\Gamma$. Moreover, from [DP, Proposition 1(c)] under this bijection and the canonical isomorphism $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \approx (\mathbb{Z}/N\mathbb{Z})^*$, $\text{Gal}(x)$ corresponds to $\text{Gal}(z) := \{k \in (\mathbb{Z}/N\mathbb{Z})^* : wz = kz \text{ for some } w \in W\}$. Thus, with the notation as in the statement of the proposition, $\mathcal{G} \subseteq \text{Gal}(x)$ if and only if $a(\alpha) \in \text{Gal}(z)$, for each $\alpha \in I$. This proves part (b) of the proposition. $\square$

For the remaining part of this section we consider the case of a *cyclic* group $\mathcal{G} = (a) \subseteq (\mathbb{Z}/N\mathbb{Z})^*$ as in (7), and we denote by

$$(9) \quad a_i \quad (1 \leq i \leq s)$$

the image of a under the canonical homomorphism $(\mathbb{Z}/N\mathbb{Z})^* \approx \prod_{i=1}^s (\mathbb{Z}/p_i^{K_i}\mathbb{Z})^* \rightarrow (\mathbb{Z}/p_i^{K_i}\mathbb{Z})^*$.

Consider the torsion $\mathbb{Z}W$ -modules

$$M = \Gamma/N\Gamma, \quad M_i = \Gamma/p_i^{K_i}\Gamma \quad (1 \leq i \leq s).$$

The canonical projections $\pi_i: M \rightarrow M_i$ ($1 \leq i \leq s$) yield an isomorphism of $\mathbb{Z}W$ -modules

$$(10) \quad \pi := \bigoplus_{i=1}^s \pi_i: M \xrightarrow{\sim} \bigoplus_{i=1}^s M_i.$$

Note that, if $z \in M$ is mapped to $(z_i)_{i=1}^s$ under this isomorphism then, given any $w \in W$, we have the equivalence

$$(11) \quad wz = az \quad \text{if and only if} \quad wz_i = a_i z_i \quad \text{for all } 1 \leq i \leq s.$$

Keeping the notation above, we define the sets

$$(12) \quad \begin{aligned} Y(a, w) &:= \{z \in M : z \text{ is reduced and } wz = az\}, \\ Y(a_i, w) &:= \{z_i \in M_i : z_i \text{ is reduced and } wz_i = a_i z_i\}. \end{aligned}$$

Lemma 1. *Notation as in (9)–(12). The isomorphism π induces a set theoretical bijection*

$$Y(a, w) \leftrightarrow \prod_{i=1}^s Y(a_i, w).$$

Proof. This follows immediately from (9)–(11). $\square$

Lemma 2. *Notation as in (9)–(12). The number of conjugacy classes of elements of $\mathfrak{G}$ of order N with $(a) \subseteq \text{Gal}(x)$ is given by*

$$\frac{1}{|W|} \sum_{w \in W} \prod_{i=1}^s |Y(a_i, w)|.$$

Proof. The argument found in the proof of [DP, Lemma 1] yields the identity

$$\sum_{w \in W} |Y(a, w)| = |W|c$$

where c is the number of W -orbits of reduced elements of M for which $wz = az$ for some $w \in W$. Now use Proposition 1(b) and Lemma 1. $\square$

2. PROOF OF THE THEOREM

In this section we consider, as in (6), a positive integer $N = \prod_{i=1}^s p_i^{K_i}$ relatively prime with $|W|$. As in (7), we fix a cyclic subgroup $\mathcal{C}$ with generator $a \in (\mathbb{Z}/N\mathbb{Z})^*$; we let n_i , $1 \leq i \leq s$, be as in (8).

Proposition 2. *The number of conjugacy classes of elements of $\mathfrak{G}$ of order N with $\mathcal{C} = (a) \subseteq \text{Gal}(x)$ is given by*

$$\frac{1}{|W|} \sum_{w \in W} \prod_{i=1}^s (p_i^{K_i f_{n_i}(w)} - p_i^{(K_i-1)f_{n_i}(w)})$$

where the $f_{n_i}(w)$ are as in (2).

Proof. This is an immediate consequence of Lemma 2 and [DP, Lemma 2] ($|Y(a_i, w)| = p_i^{K_i f_{n_i}(w)} - p_i^{(K_i-1)f_{n_i}(w)}$, since n_i is the order of a_i in $(\mathbb{Z}/p_i^{K_i}\mathbb{Z})^*$). $\square$

We can reformulate Proposition 2 in terms of the polynomial G defined in (3) and (4).

Proof of the Theorem. Notation as above. We have the equalities

$$\begin{aligned}
 & \prod_{i=1}^s (p_i^{K_i f_{n_i}(w)} - p_i^{(K_i-1)f_{n_i}(w)}) \\
 &= \sum_{\varepsilon=(\varepsilon_1, \dots, \varepsilon_s) \in (\mathbb{Z}/2\mathbb{Z})^s} (-1)^{|\varepsilon|} p_1^{(K_1-\varepsilon_1)f_{n_1}(w)} \dots p_s^{(K_s-\varepsilon_s)f_{n_s}(w)} \\
 &= \sum_{\varepsilon=(\varepsilon_1, \dots, \varepsilon_s) \in (\mathbb{Z}/2\mathbb{Z})^s} (-1)^{|\varepsilon|} \prod_{j=1}^n \left(\prod_{i: n_i=j} p_i^{(K_i-\varepsilon_i)} \right)^{f_j(w)} \\
 &= \sum_{\varepsilon=(\varepsilon_1, \dots, \varepsilon_s) \in (\mathbb{Z}/2\mathbb{Z})^s} (-1)^{|\varepsilon|} q_1(\varepsilon)^{f_1(w)} \dots q_h(\varepsilon)^{f_h(w)}.
 \end{aligned}$$

The theorem now follows from Proposition 2. $\square$

Corollary 1 (Djoković [Djk1]). *Let N be coprime with $|W|$. The number of conjugacy classes of elements of $\mathfrak{G}$ of exponent N is*

$$\nu(N) = \prod_{i=1}^l \frac{m_i + N}{m_i + 1}$$

where $m_1 \leq m_2 \leq \dots \leq m_l$ are the exponents of the Weyl group W of $\mathfrak{G}$.

Proof. We first evaluate the number of conjugacy classes of elements of $\mathfrak{G}$ of order N with $(1) \subseteq \text{Gal}(x)$ by looking at the particular case $n_1 = n_2 = \dots = n_s = 1$ in (8). From the Theorem this number equals

$$\sum_{\varepsilon=(\varepsilon_1, \dots, \varepsilon_s) \in (\mathbb{Z}/2\mathbb{Z})^s} (-1)^{|\varepsilon|} G(q_1(\varepsilon), \dots, q_h(\varepsilon))$$

where here $q_1(\varepsilon) = \prod_{i=1}^s p^{(K_i-\varepsilon_i)}$, $q_j(\varepsilon) = 1$ ($j \neq 1$), and $|\varepsilon| = \sum_{i=1}^s \varepsilon_i$.

For a fixed ε , we have

$$\begin{aligned}
 G(q_1(\varepsilon), \dots, q_h(\varepsilon)) &= \frac{1}{|W|} \sum_{k_1, \dots, k_h \in \mathbb{N}} a_{k_1, \dots, k_h} q_1(\varepsilon)^{k_1} \\
 &= \frac{1}{|W|} \sum_{k_1} \left(\sum_{k_2, \dots, k_h} a_{k_1, \dots, k_h} \right) q_1(\varepsilon)^{k_1}.
 \end{aligned}$$

Note that $\sum_{k_2, \dots, k_h} a_{k_1, \dots, k_h}$ is the number of elements w of W with 1 as an eigenvalue of multiplicity k_1 , which we will denote $s_{k_1}(1)$ (as in [PW]). Thus,

$$G(q_1(\varepsilon), \dots, q_h(\varepsilon)) = \frac{1}{|W|} \sum_{k_1} s_{k_1}(1) q_1(\varepsilon)^{k_1}.$$

But, from [Slm], the right-hand side of this equality is given by $\prod_{i=1}^l (m_i + q_1(\varepsilon)) / (m_i + 1)$ where $m_1, \dots, m_l$ are the exponents of W .

As an intermediate step we therefore have the following

Corollary 2. *The number of conjugacy classes of elements of $\mathfrak{G}$ of order N is given by*

$$(13) \quad \sum_{\varepsilon=(\varepsilon_1, \dots, \varepsilon_s) \in (\mathbb{Z}/2\mathbb{Z})^s} (-1)^{|\varepsilon|} \nu(q_1(\varepsilon))$$

where $\nu(t) := \prod_{i=1}^l (m_i + t) / (m_i + 1)$. $\square$

We now finish the proof of Corollary 1. Assume Djoković's formula is true for any $M < N$ (it is obviously true for $N = 1$). Then, using an "inclusion-exclusion" argument, the number of conjugacy classes of elements of exponent N but order less than N is easily seen to be

$$(14) \quad \sum_{(0, \dots, 0) \neq \varepsilon \in (\mathbb{Z}/2\mathbb{Z})^s} (-1)^{|\varepsilon|-1} \nu(q_1(\varepsilon)).$$

Combining (13) and (14), we conclude that the total number of conjugacy classes of elements of exponent N is equal to $\nu(q_1(0))$, which is $\nu(N)$, as was to be shown. $\square$

Next, we show how [DP, Theorem 1] follows from the Theorem.

We fix a prime number p not dividing $|W|$, and an integer $K \geq 1$. We set $N = p^K$. So, adopting the notation of (6), we have $s = 1$. We fix a subgroup of $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \approx (\mathbb{Z}/p^K\mathbb{Z})^*$, necessarily cyclic, say (a) . We set $n = \text{ord}(a)$ as in (8). So, here, $n_1 = n$.

If x is any element of $\mathfrak{G}$ of order $N = p^K$, then its Galois group $\text{Gal}(x)$ is a subgroup of the cyclic group $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, and hence is determined by its order, called the *depth* of x .

[DP, Theorem 1(a)] counts the number of conjugacy classes of elements of order p^K , with depth divisible by n , i.e., in the present context, with $(a) \subseteq \text{Gal}(x)$. In view of Proposition 1(a) (or [DP, Proposition 2(a)]), we may assume that $n | \gcd(p-1, e(w))$, as in the hypothesis of [DP, Theorem 1(a)].

In [DP, PW], the polynomials used to do this counting, are

$$A_n(t) = -1 + \prod_{i \in I(n)} \frac{m_i + t}{m_i + 1}$$

where $m_1 \leq m_2 \leq \dots \leq m_l$ are the exponents of the Weyl group W of $\mathfrak{G}$, and where $I(n) := \{1 \leq i \leq l : n \text{ divides } m_i + 1\}$. But from [PW, Theorem 2], we have

$$A_n(t) = \frac{1}{|W|} \sum_{w \in W} t^{f_n(w)} - 1,$$

i.e.,

$$(15) \quad A_n(t) = G(t_1, \dots, t_h) - 1$$

with

$$\begin{aligned} t_n &:= t & \text{if } 1 \leq n \leq h, \\ t_j &:= 1 & \text{for any } 1 \leq j \neq n \leq h. \end{aligned}$$

Finally, the expression

$$\sum_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^s} (-1)^{|\varepsilon|} G(q_1(\varepsilon), \dots, q_h(\varepsilon))$$

of the Theorem, now reads as

$$(16) \quad \sum_{\varepsilon=0,1} (-1)^\varepsilon G(q_1(\varepsilon), \dots, q_h(\varepsilon))$$

where

$$\begin{aligned} q_n(\varepsilon) &= p^{K-\varepsilon} & \text{if } 1 \leq n \leq h, \\ q_j(\varepsilon) &= 1 & \text{for any } 1 \leq j \leq h, \quad j \neq n. \end{aligned}$$

From (15), we rewrite (16) as $A_n(p^K) - A_n(p^{K-1})$. All together, we have derived the following result from the Theorem.

Corollary 3 [DP, Theorem 1]. *Let p be a prime number not dividing $|W|$. Let $n|\gcd(p-1, e(w))$. The number of conjugacy classes of elements of $\mathfrak{G}$ of order p^K , with $p^K \geq 1$, and depth divisible by n is $A_n(p^K) - A_n(p^{K-1})$.*

N.b. We have derived Theorem 1 of [DP] from the Theorem above, but note that the proof of the present theorem makes use of the crucial lemmas and propositions in [DP].

3. A REMARK ON COMPUTATIONS

The polynomial $G(t_1, \dots, t_h)$ for a particular Weyl group can explicitly be computed using the work of Carter [Ctr], where conjugacy classes of the Weyl group W of $\mathfrak{G}$ are described by certain graphs (or Dynkin diagrams) denoted in [Ctr] by Γ (not to be confused with our notation in (1)!) (see [Ctr, pp. 24–31, for classical groups and pp. 31–34 and pp. 48–58 for the exceptional ones]). Also, one can find in [Ctr], for each graph Γ , the cardinality $|\text{ccl}_W(w)|$ of the conjugacy class of the corresponding element w of W (see [Ctr, pp. 34–45, 48–58]).

Finally, the characteristic polynomial $p_w(t)$ of an element w of W is computed from a graph Γ representing its conjugacy class, using [Ctr, Table 3, p. 23], via a straightforward algorithm [Ctr, pp. 19–24]. Knowing $p_w(t)$, one can then deduce easily the multiplicity $f_j(w)$ of ζ_j as an eigenvalue of w , for each $1 \leq j \leq h$.

We then use the expression

$$(17) \quad G(t_1, \dots, t_h) = \frac{1}{|W|} \sum_{\Gamma} |\text{ccl}_W(w)| t_1^{f_1(w)} \dots t_h^{f_h(w)},$$

which follows from (4) using the fact that the monomials $t_1^{f_1(w)} \dots t_h^{f_h(w)}$ are invariant under conjugation in W .

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