FINITE DIMENSIONALITY OF IRREDUCIBLE UNITARY REPRESENTATIONS OF COMPACT QUANTUM GROUPS

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Abstract. In this paper, we study the representations of Hopf C*-algebras; the main result is that every irreducible left unitary representation of a Hopf C*-algebra with a Haar measure is finite dimensional. To prove this result, we first study the comodule structure of the space of Hilbert-Schmidt operators; then we use this comodule structure to show that every irreducible left unitary representation of a Hopf C*-algebra with a Haar measure is finite dimensional.

1. Introduction

It is well known that, for compact groups, every irreducible unitary representation is finite dimensional. For a simple proof about this result, we refer to [N]. In this paper, we will generalize this result to Hopf C*-algebras with Haar measures; namely, we will prove that every irreducible left unitary representation of a Hopf C*-algebra with a Haar measure is finite dimensional. To prove this result, we first study the comodule structure of Hilbert-Schmidt operators; then we use this comodule structure to show that every irreducible left unitary representation is finite dimensional.

In earlier work in this direction, Woronowicz [W] proved that every irreducible unitary representation of a compact matrix quantum group is finite dimensional; the author [Q] showed that for a Hopf C*-algebra with the Peter-Weyl property, every irreducible unitary representation is finite dimensional. The previous approach depends heavily on the Peter-Weyl property. Here we are going to generalize these results; the method we use is elementary, which does not use the Peter-Weyl property.

Before we turn to the contents of the paper, let us recall the definitions of Hopf C*-algebras, representations, and Haar measures of Hopf C*-algebras.

Let A be a C*-algebra with a dense *-subalgebra \( \mathcal{A} \) and \( \Phi: A \to A \otimes A \) a
\( C^* \)-homomorphism. We say that \((A, \Phi)\) is a Hopf \( C^* \)-algebra if the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\Phi} & A \otimes A \\
\downarrow & & \downarrow \Phi \otimes 1 \\
A \otimes A & \xrightarrow{1 \otimes \Phi} & A \otimes A \otimes A \\
\end{array}
\]

commutes and \( \Phi(\mathcal{A}) \subset \mathcal{A} \otimes \mathcal{A} \), where \( \Phi \) is the comultiplication of \((A, \Phi)\).

If \( A \) is a von Neumann algebra, we call \( A \) a Hopf-von Neumann algebra. By an involution of \((A, \Phi)\), we mean a *-anti-isomorphism \( k: \mathcal{A} \to \mathcal{A} \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{k} & \mathcal{A} \\
\downarrow & & \downarrow \tau \\
\mathcal{A} \otimes \mathcal{A} & \xrightarrow{k \otimes k} & \mathcal{A} \otimes \mathcal{A} \\
\end{array}
\]

commutes where \( \tau: A \otimes A \to A \otimes A \) is the flip automorphism and \( \tau(a \otimes b) = b \otimes a, \forall a, b \in A \).

We say that \( e: \mathcal{A} \to C \) is a counit of \( A \) if \( e \) is a \( C^* \)-homomorphism and the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{A} \otimes C & \xrightarrow{1 \otimes e} & \mathcal{A} \\
\downarrow & & \downarrow \Phi \\
\mathcal{A} \otimes \mathcal{A} & \xrightarrow{e \otimes 1} & C \otimes \mathcal{A} \\
\end{array}
\]

We say that \((A, \Phi, k, e)\) is a compact quantum group if the following diagrams commute:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{e} & C & \xrightarrow{i} & \mathcal{A} \\
\downarrow & & \downarrow m \\
\mathcal{A} \otimes \mathcal{A} & \xrightarrow{k \otimes 1} & \mathcal{A} \otimes \mathcal{A} \\
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{e} & C & \xrightarrow{i} & \mathcal{A} \\
\downarrow & & \downarrow m \\
\mathcal{A} \otimes \mathcal{A} & \xrightarrow{1 \otimes k} & \mathcal{A} \otimes \mathcal{A} \\
\end{array}
\]

Let \( A \) be a coalgebra and \( M \) a linear space. Let \( \psi: M \to A \otimes M \) be a linear
map which makes the following diagrams commute:

\[ \begin{align*}
C \otimes M & \xrightarrow{e \otimes I} A \otimes M \\
& \xrightarrow{\otimes \psi} M \\
A \otimes A \otimes M & \xrightarrow{I \otimes \psi} A \otimes M \\
& \xrightarrow{\Delta \otimes I} M \\
A \otimes M & \xrightarrow{\psi} M
\end{align*} \]

The pair \((M, \psi)\) is called a unital left \(A\)-comodule; if only the second diagram commutes, we call \((M, \psi)\) a left \(A\)-comodule, and \(\psi\) is said to be its structure map. A right \(A\)-comodule can be defined similarly. For more information about comodules, we refer to [A].

Let \(M\) be a left \(A\)-comodule with structure map \(\psi\); a subspace \(M_1 \subset M\) is said to be left invariant if \(\psi(M_1) \subset A \otimes M_1\). We say that \(M\) is an irreducible left \(A\)-comodule if \(M_1\) is the only nonzero left invariant subspace of \(M\).

Now let \((A, \Phi)\) be a Hopf \(C^*\)-algebra with unit and \(\mathcal{A}\) the dense \(*\)-subalgebra of \(A\). We say that \(V\) is a left \(A\)-comodule if it is a left \(\mathcal{A}\)-comodule which is defined as above. Suppose that \(V\) is a finite-dimensional left \(A\)-comodule with structure map \(\psi: V \to \mathcal{A} \otimes V\). If \(\{e_i\}_{i=1}^n\) is a basis of \(V\) and

\[ \psi(e_i) = \sum_{k=1}^n a_{ik} \otimes e_k, \]

the matrix \((a_{ij})\) is called the coefficient matrix of \(\psi\) with respect to \(\{e_i\}_{i=1}^n\). Then the comodule property implies that

\[ \Phi(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}. \]

Now suppose that \(V\) is a left \(A\)-comodule, which is also a Hilbert space, with structure map \(L: V \to \mathcal{A} \otimes V\) and inner product \((\cdot, \cdot): V \times V \to C\). We can extend \((\cdot, \cdot)\) to \((\cdot, \cdot)\): \((\mathcal{A} \otimes V) \times (\mathcal{A} \otimes V) \to \mathcal{A}\) as

\[ \langle a \otimes x, b \otimes y \rangle = ab^*(x, y), \quad \forall a, b \in \mathcal{A}, \; x, y \in V. \]

A left unitary representation \(\pi\) of \(A\) on a Hilbert space \(H\) is a comodule map from \(H\) into \(\mathcal{A} \otimes H\) such that

\[ \langle \pi(x), \pi(y) \rangle = \langle x, y \rangle, \quad \forall x, y \in H. \]

Let \((A, \Phi, k, e)\) be a Hopf \(C^*\)-algebra and \(\sigma\) a positive linear functional. We say that \(\sigma\) is a left Haar measure if, for all \(x^* \in A^*\), we have

\[ x^* \cdot \sigma = (x^*, I) \sigma. \]

Similarly, we can define a right Haar measure.

Note that for a left (right) Haar measure \(\sigma\), we have \(\sigma(k(a)) = \sigma(a), \; \forall a \in \mathcal{A}\). For a proof of this result, we refer to [W].
Finally, we come to the contents of this paper. In §2 we study the comodule structure of Hilbert-Schmidt operators. The main results in this section are that, for any two Hilbert spaces $H_1, H_2$, which are left unitary left $\mathcal{A}$-comodules, where $\mathcal{A}$ is the dense $*$-subalgebra of a Hopf $C^*$-algebra, the space $\text{Hom}_2(H_1, H_2)$ of Hilbert-Schmidt operators from $H_1$ to $H_2$ has a natural right $\mathcal{A}$-comodule structure. Also we give a characterization that under what condition a Hilbert-Schmidt operator is a comodule map. In §3 we first show that the space of Hilbert-Schmidt operators is invariant under the action of a Haar measure. Then we use the results in §2 to show that every irreducible left unitary representation of a Hopf $C^*$-algebra with a Haar measure is finite dimensional.

2. Comodule structure for Hilbert-Schmidt operators

In this section, we are going to endow a comodule structure to the space of Hilbert-Schmidt operators between two Hilbert spaces which are left unitary comodules of a Hopf $C^*$-algebra.

Let $H_1, H_2$ be Hilbert spaces. An operator $T \in B(H_1, H_2)$ is said to be a Hilbert-Schmidt operator if, for one orthonormal basis $(e_j)$ of $H_1$, $\sum_i \|Te_j\|^2 < \infty$. Let $\text{Hom}_2(H_1, H_2)$ denote the space of all Hilbert-Schmidt operators from $H_1$ to $H_2$. For every $T \in \text{Hom}_2(H_2, H_2)$, let

$$\|T\|_2^2 = \sum_i \|Te_j\|^2.$$ 

It is well known that $(\text{Hom}_2(H_1, H_2), \|\cdot\|_2)$ is a Hilbert space with the inner product given by

$$\langle T_1, T_2 \rangle = \sum_i \langle T_1e_i, T_2e_i \rangle.$$ 

Now let $(A, \Phi, k, e)$ be a Hopf $C^*$-algebra with a dense $*$-subalgebra $\mathcal{A}$. Suppose that $H_1, H_2$ are also left unitary $\mathcal{A}$-comodules with structure maps $\psi_1$ and $\psi_2$. Fix orthonormal bases $(e_i^1)$, $(e_i^2)$ for $H_1, H_2$ respectively. Suppose that

$$\psi_1(e_i^1) = \sum_k a_{ik}^1 \otimes e_k^1, \quad \psi_2(e_i^2) = \sum_s a_{s}^2 \otimes e_s^2.$$ 

Let $H_1'$ be the complex conjugate of $H_1$. Then $(e_i^1)$ also form an orthonormal basis for $H_1'$. It is straightforward to verify that

$$\psi_1(e_i^1) = \sum_k e_k^1 \otimes a_{ki}^1$$

gives $H_1'$ a right $\mathcal{A}$-comodule structure.

Since $\text{Hom}_2(H_1, H_2) \cong H_1' \otimes H_2$ (projective tensor product), define

$$\psi_{H_1, H_2}: \text{Hom}_2(H_1, H_2) \rightarrow \text{Hom}_2(H_1, H_2) \otimes \mathcal{A}$$

as

$$\psi_{H_1, H_2}(e_i^1 \otimes e_j^2) = \sum_{k, s} (e_k^1 \otimes e_s^2) \otimes a_{k}^1 \otimes (a_{js}^2).$$
Proposition 2.1. With the notation as above, \((\text{Hom}_2(H_1, H_2), \psi_{H_1, H_2})\) is a right \(\mathcal{A}\)-comodule with structure map \(\psi_{H_1, H_2}\).

Proof. We only need to verify that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Hom}_2(H_1, H_2) & \xrightarrow{\psi_{H_1, H_2}} & \text{Hom}_2(H_1, H_2) \otimes \mathcal{A} \\
\downarrow \psi_{H_1, H_2} & & \downarrow \psi_{H_1, H_2} \otimes I \\
\text{Hom}_2(H_1, H_2) \otimes \mathcal{A} & \xrightarrow{I \otimes \Phi} & \text{Hom}_2(H_1, H_2) \otimes \mathcal{A} \otimes \mathcal{A}
\end{array}
\]

In fact, for \(e_1^i \otimes e_2^j\), we have

\[
(\psi_{H_1, H_2} \otimes I)\psi_{H_1, H_2}(e_1^i \otimes e_2^j) = \sum_{k,s} \psi_{H_1, H_2}(e_k^i \otimes e_s^j) \otimes a_{ki}^1 k(a_{js}^2)
\]

\[
= \sum_{k,s} \sum_{p,q} (e_p^i \otimes e_q^j) \otimes a_{pk}^1 k(a_{sq}^2) \otimes a_{ki}^1 k(a_{js}^2).
\]

On the other hand, we have

\[
(I \otimes \Phi)\psi_{H_1, H_2}(e_1^i \otimes e_2^j) = \sum_{k,s} (e_k^i \otimes e_s^j) \Phi(a_{ki}^1 k(a_{js}^2))
\]

\[
= \sum_{k,s} \sum_{p,q} (e_p^i \otimes e_q^j) \otimes a_{pk}^1 k(a_{sq}^2) \otimes a_{ki}^1 k(a_{js}^2).
\]

This completes the proof.

For \(T \in \text{Hom}_2(H_1, H_2)\), \(v \in H_1\), we have

\[
(\psi_{H_1, H_2} T)(v) = \tau(m \otimes I)(I \otimes k \otimes I)(I \otimes \psi_2)(I \otimes T)(\psi_1(v)).
\]

The main result of this section is

Theorem 2.2. With the above assumption and for \(T \in \text{Hom}_2(H_1, H_2)\), the following are equivalent:

1. \(T\) is a left \(\mathcal{A}\)-comodule map,
2. \(T\) is a right \(A^*\)-module map,
3. \(\psi_{H_1, H_2}(T) = T \otimes I\),
4. \(x^* \cdot T = x^*(I)T\), \(\forall x^* \in A^*\).

Proof. The equivalence between (1) and (2) and (3) and (4) is well known. So we only need to prove the equivalence of (1) and (3).

(1) \(\Rightarrow\) (3). Suppose that \(T\) is a left \(\mathcal{A}\)-comodule map. Then

\[
\begin{array}{ccc}
H_1 & \xrightarrow{T} & H_2 \\
\downarrow \psi_1 & & \downarrow \psi_2 \\
\mathcal{A} \otimes H_1 & \xrightarrow{I \otimes T} & \mathcal{A} \otimes H_2
\end{array}
\]

commutes. So, for any \(e_1^i \in H_1\), we have

\[
\psi_2(T e_1^i) = (I \otimes T) \psi_1(e_1^i),
\]
\[ \psi_2(Te_1^1) = \sum_k (I \otimes T)(a_{ik}^1 \otimes e_k^1) = \sum_k a_{ik}^1 \otimes Te_k^1, \]

\[ (\psi_{H_1,H_2}(T))(e_1^1) = \tau(m \otimes I)(I \otimes k \otimes I)(I \otimes \psi_2)(I \otimes T)\psi_1(e_1^1) \]

\[ = \tau(m \otimes I)(I \otimes k \otimes I)(I \otimes \psi_2) \left( \sum_k a_{ik}^1 \otimes Te_k^1 \right) \]

\[ = \tau(m \otimes I)(I \otimes k \otimes I) \left( \sum_{k,s} a_{ik}^1 \otimes a_{ks}^1 \otimes Te_s^1 \right) \]

\[ = \sum_{k,s} Te_s^1 \otimes a_{ik}^1 k(a_{ks}^1) = \sum_s Te_s^1 \otimes e(a_{is}^1) \]

\[ = T \left( \sum_s e(a_{is}^1)e_s^1 \right) \otimes I = T(e_s^1) \otimes I. \]

So \( \psi_{H_1,H_2}(T) = T \otimes I. \)

(3) \( \Rightarrow \) (1). Suppose that \( \psi_{H_1,H_2}(T) = T \otimes I. \) Then, for any \( e_i^1 \in H_1, \) we have

\[ (\psi_{H_1,H_2}(T))(e_i^1) = T(e_i^1) \otimes I. \]

Let \( Te_i^1 = \sum_p b_{ip}e_p^2. \) So we have

\[ (\psi_{H_1,H_2}(T))(e_i^1) = \tau(m \otimes I)(I \otimes k \otimes I)(I \otimes \psi_2) \left( \sum_k a_{ik}^1 \otimes Te_k^1 \right) \]

\[ = \tau(m \otimes I)(I \otimes k \otimes I) \left( \sum_k a_{ik}^1 \otimes \psi_2(\psi_1)(e_i^1) \right) \]

\[ = \sum_{k,p} b_{kp} \tau(m \otimes I)(I \otimes k \otimes I) \left( a_{ik}^1 \otimes \sum_s a_{ps}^2 \otimes e_s^2 \right) \]

\[ = \sum_{k,p,s} b_{kp} e_s^2 \otimes a_{ik}^1 k(a_{ps}^2). \]

Thus

\[ (m \otimes I)(2, 3, 1)(\psi_2 \otimes I)(\psi_{H_1,H_2}(T))(e_i^1) \]

\[ = (m \otimes I)(2, 3, 1) \left( \sum_{k,p,s,t} b_{kp} a_{st}^2 \otimes e_t^2 \otimes a_{ik}^1 k(a_{ps}^2) \right) \]

\[ = (m \otimes I) \left( \sum_{k,p,s,t} b_{kp} a_{ik}^1 k(a_{ps}^2) a_{st}^2 \otimes e_t^2 \right) = \sum_{k,p,s,t} b_{kp} a_{ik}^1 k(a_{ps}^2) a_{st}^2 \otimes e_t^2 \]

\[ = \sum_{k,p,t} b_{kp} e_t^2 a_{ik}^1 \otimes e_t^2 = \sum_{k,p} a_{ik}^1 \otimes b_{kp} e_p^2 = \sum_{k} a_{ik}^1 \otimes Te_i^1, \]

where \( (2, 3, 1) \) is the map from \( \mathcal{A} \otimes H \otimes \mathcal{A} \) into \( \mathcal{A} \otimes \mathcal{A} \otimes H \) such that
Thus we have that $\psi_2(Te_i^1) = \sum_k a_{ik}^1 \otimes Te_k^1$, i.e., $T$ is a comodule map. This completes the proof.

3. Finite dimensionality of irreducible unitary representations

In this section, we are going to show that every irreducible unitary representation of a Hopf $C^*$-algebra with a Haar measure is finite dimensional. We begin the section with the following result, which is the consequence of Theorem 2.2.

Proposition 3.1. With the same notation as above, if $T \in \text{Hom}_2(H_1, H_2)$, then $\sigma \cdot T \in \text{Hom}_2(H_1, H_2)$.

Proof. Since $T = \sum_{i,j} b_{ij} e_i^1 \otimes e_j^2$, where $\sum_{i,j} |b_{ij}|^2 < \infty$,

$$\sigma \cdot T = (I \otimes \sigma) (H_1, H_2)(T)$$

$$= (I \otimes \sigma) \left[ \sum_{i,j} b_{ij} \psi_{H_1, H_2}(e_i^1 \otimes e_j^2) \right]$$

$$= (I \otimes \sigma) \left[ \sum_{i,j} b_{ij} \sum_{k,s} (e_k^1 \otimes e_s^2) a_{ki}^1 a_{js}^1 \right]$$

$$= \sum_{k,s} \left[ \sum_{i,j} b_{ij} \sigma(a_{ki}^1 a_{js}^1) \right] e_k^1 \otimes e_s^2.$$ 

But

$$\sum_{k,s} \sum_{i,j} |b_{ij}|^2 |\sigma(a_{ki}^1 a_{js}^1)|^2 \leq \sum_{k,s} \sum_{i,j} |b_{ij}|^2 \sigma(a_{ki}^1 (a_{ki}^1)^*) \sigma(a_{js}^1 (a_{js}^1)^*)$$

$$= \sum_{i,j} |b_{ij}|^2 \sigma \left( \sum_k a_{ki}^1 (a_{ki}^1)^* \right) \sigma \left( \sum_s a_{js}^1 (a_{js}^1)^* \right) = \sum_{i,j} |b_{ij}|^2,$$

since $H_1, H_2$ are unitary $\mathcal{A}$-comodules. Hence $\sigma \cdot T \in \text{Hom}_2(H_1, H_2)$. This finishes the proof.

Lemma 3.2. If $\sigma$ is a left Haar measure on $\mathcal{A}$, then, for any $T \in \text{Hom}_2(H_1, H_2)$, $\sigma \cdot T$ is a left $\mathcal{A}$-comodule map.

Proof. It is a direct consequence of Theorem 2.2 and the definition of a Haar measure.

Theorem 3.3. If $(A, \Phi, k, e)$ is a Hopf $C^*$-algebra with a dense subalgebra $\mathcal{A}$ and a Haar measure $\sigma$, then every irreducible left unitary representation is finite dimensional.
Proof. Suppose that $H$ is an irreducible left unitary $\mathcal{A}$-comodule with structure map $\pi$. Choose an orthonormal basis $\{e_i\}_{i \in I}$ for $H$. Let $\pi(e_i) = \sum_j a_{ij} \otimes e_j$. Then, for any $i, j, p, q \in I$, we have
\[
\langle e_i, \sigma \cdot (e_p \otimes e_q)e_j \rangle = \langle e_i, (I \otimes \sigma)\psi_{H,H}(e_p \otimes e_q)e_j \rangle \\
= \left\langle e_i, (I \otimes \sigma) \sum_{k,s} (e_k \otimes e_s)a_{pk}k(a_{qs}) e_j \right\rangle \\
= \sum_{k,s} \sigma(a_{pk}k(a_{qs}))(e_j, e_k)(e_s, e_i) \\
= \sigma(a_{pj}k(a_{qi})).
\]
Since $\pi$ is a left unitary representation, there exist $i, j, p, q \in I$ such that $\sigma(a_{pj}k(a_{qi})) \neq 0$. Also, $e_p \otimes e_q \in \text{Hom}_2(H, H)$, so $\sigma \cdot (e_p \otimes e_q) \in \text{Hom}_2(H, H)$. Because $\sigma \cdot (e_p \otimes e_q)$ is a left comodule map and $H$ is irreducible, there exists $0 \neq \lambda \in \mathbb{C}$ such that
\[
\sigma \cdot (e_p \otimes e_q) = \lambda \cdot I.
\]
Thus we get that $I \in \text{Hom}_2(H, H)$. This implies that $\dim H < \infty$. This finishes the proof.

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References


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