FUNCTIONS WITH A UNIQUE MEAN VALUE AND AMENABILITY

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Abstract. It is shown that there exist many amenable locally compact groups for which the sets of functions with unique left invariant mean values are not closed under addition. This resolves negatively a problem raised by T. Miao.

Let $G$ be a locally compact group with a fixed left Haar measure $\lambda$ and let $L^p(G)$ ($1 \leq p \leq \infty$) be the associated Lebesgue spaces. (As usual, if $G$ is compact, then $\lambda$ is assumed to be normalized, i.e., $\lambda(G) = 1$.) A subspace $S$ of $L^\infty(G)$ is said to be admissible if it contains the constants and $x_f$ for each $f \in S$ and $x \in G$ (where $x_f$ is the left translate of $f$ by $x$, $x_f(y) = f(xy)$ ($y \in G$)). If $f \in L^\infty(G)$, let $S_f$ denote the smallest admissible subspace containing $f$. We say that $f \in L^\infty(G)$ has a unique left invariant mean value if $\text{LIM}(S_f)$ is nonempty and there exists a constant $c$ such that $m(f) = c$ for each $m \in \text{LIM}(S_f)$. (For an admissible subspace $S$ of $L^\infty(G)$, $\text{LIM}(S)$ stands for the set of left invariant means on $S$, i.e., all $m \in S^*$ with $m \geq 0$, $m(1) = 1$, and $m(x_f) = m(f)$ ($x \in G, f \in S$).) The set of functions with a unique left invariant mean value is denoted by $U(G)$. Note that $U(G)$ is always closed under scalar multiplication. Recall also that if $G$ is amenable as a discrete group, then $U(G)$ coincides with the sum of the constants and the norm closed linear span of $\{f - x_f : f \in L^\infty(G), x \in G\}$ (see [6, Theorem 1.1]). In particular, $U(G)$ is closed under addition for such a $G$. Recently Miao [3, Theorem 3.4] proved that if $U(G)$ is closed under addition, then $G$ is amenable, thus answering a question raised by Rosenblatt and Yang in [6, p. 747]. The following was posed as an open problem in [3, p. 1083]: Is $U(G)$ closed under addition if $G$ is amenable? The purpose of the present note is to give many examples which show that the answer to Miao’s problem is negative.

Recall that a compact group $G$ is said to have the mean zero weak containment property if there exists a net $\{g_\alpha\}$ in

$$L^2_0(G) = \left\{f \in L^2(G) : \int_G f \, d\lambda = 0 \right\}$$

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such that \( \|g_\alpha\|_2 = 1 \) for all \( \alpha \) and \( \lim_{\alpha} \|x g_\alpha - g_\alpha\|_2 = 0 \) for all \( x \in G \) (see [5]).

The negative answer to Miao's problem is a direct consequence of the following result.

**Theorem.** Let \( G \) be an infinite compact group and suppose that \( U(G) \) is closed under addition. Then \( G \) has the mean zero weak containment property.

**Proof.** Assume, by way of contradiction, that \( G \) does not have the mean zero weak containment property. We denote by \( C \) and \( H \) the constant functions on \( G \) and the linear span of \( \{f - xf: f \in L^\infty(G), \, x \in G\} \), respectively. Let us first show that \( H + C \) is included in \( U(G) \). Obviously, \( C \) is contained in \( U(G) \). Let \( h \) be in \( \{f - xf: f \in L^\infty(G), \, x \in G\} \). Since \( \lambda \) induces a left invariant mean on \( S_h \), \( \operatorname{lim}(S_h) \) is nonempty. Observe that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k (f - x f) = 0
\]

for each \( f \in L^\infty(G) \) and \( x \in G \). Using this fact, we can see that \( m(h) = 0 \) for each \( m \in \operatorname{lim}(S_h) \). Thus \( h \) has a unique left invariant mean value. As \( U(G) \) is closed under addition, \( U(G) \) contains \( H + C \), as desired. On the other hand, since \( G \) does not have the mean zero weak containment property, it follows from the proofs of [8, Lemma and Proposition 2] that \( L^\infty(G) = H + C \). In conclusion, we obtain \( L^\infty(G) = U(G) \). But this is a contradiction. In fact, let \( E \) be an open dense subset of \( G \) satisfying \( \lambda(E) < 1 \). (It is possible to find such an \( E \) because \( G \) is an infinite compact group.) Then there exists a left invariant mean \( m \) on \( S_{x_E} \) such that \( m(\chi_E) = 1 \), where \( \chi_E \) denotes the characteristic function of \( E \) (cf. [3, Lemma 3.1]). This means that both 1 and \( \lambda(E) \) \( (\leq 1) \) can be left invariant mean values of \( \chi_E \) on \( S_{x_E} \). We therefore have \( \chi_E \not\in U(G) \), which gives the desired contradiction. The proof is now complete.

It is well known that if an infinite compact group \( G \) is amenable as a discrete group, then \( G \) has the mean zero weak containment property (cf. [4, Theorem 1.3 and Lemma 3.1]). Some examples of compact groups which do not have the mean zero weak containment property can be found in [1, 2, 5, 7]. For example, \( \text{SO}(n) \) (the special orthogonal group) does not have the mean zero weak containment property for \( n \geq 3 \). Therefore our Theorem implies that the set of functions on \( \text{SO}(n) \) \( (n \geq 3) \) with a unique left invariant mean value is not closed under addition. This resolves Miao's problem [3, p. 1083] negatively.

As a consequence of our Theorem we have the following. This gives a number of examples which show that the answer to Miao's problem is negative.

**Proposition.** Let \( G_1 \) be an infinite compact group that does not have the mean zero weak containment property and let \( G_2 \) be an amenable locally compact group. Then \( G_1 \times G_2 \) is an amenable locally compact group for which \( U(G_1 \times G_2) \) is not closed under addition.

**Proof.** Let us find two functions in \( U(G_1 \times G_2) \) whose sum does not have a unique left invariant mean value. Since, by the Theorem, \( U(G_1) \) is not closed under addition, there exist \( h \) and \( k \) in \( U(G_1) \) such that \( h + k \) is not contained in \( U(G_1) \). Now define functions \( f \) and \( g \) in \( L^\infty(G_1 \times G_2) \) by

\[
f(x, y) = h(x) \quad \text{and} \quad g(x, y) = k(x) \quad ((x, y) \in G_1 \times G_2).
\]
Then \( f \) and \( g \) have unique left invariant mean values but \( f+g \) does not. This can be easily verified from [6, Proposition 1.3] and a straightforward argument. Thus we obtain that \( U(G_1 \times G_2) \) is not closed under addition.

Remarks. (1) It is possible that a compact group \( G \) has the mean zero weak containment property, and \( U(G) \) is not closed under addition. For example, take \( G = \text{SO}(n) \times \mathbb{T} \quad (n \geq 3) \), where \( \mathbb{T} \) denotes the circle group. Then the Proposition implies that \( U(G) \) is not closed under addition. Since \( L^\infty(G) \) has more than one left invariant mean, it follows from [4, Theorem 1.3 and Lemma 3.1] that \( G \) has the mean zero weak containment property. Thus the converse of our Theorem is false.

(2) Recall Miao’s Theorem [3, Theorem 3.4] quoted above: If \( U(G) \) is closed under addition, then \( G \) is amenable. Our Theorem and Proposition may be regarded as partial improvements of this result. It would be of interest to determine the class of locally compact groups \( G \) for which \( U(G) \) is closed under addition.

(3) Let \( H \) be as in the proof of our Theorem. It is worthwhile to observe the following: If \( G \) is a locally compact group and if \( U(G) \) is closed under addition, then

\[
H + \mathbb{C} \subseteq U(G) \subseteq \overline{H} + \mathbb{C} \quad (= \overline{H + \mathbb{C}}),
\]

where the over bar denotes the norm closure in \( L^\infty(G) \). The argument used in the proof of the Theorem can apply to show the first inclusion relation. (Notice that the closure of \( U(G) \) under addition implies amenability of \( G \) [3, Theorem 3.4].) The second inclusion relation follows from the proof of [3, Theorem 3.6].

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