PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 3, July 1994

WHAT STRONG MONOTONICITY CONDITION ON FOURIER COEFFICIENTS CAN MAKE THE RATIO $\|f - S_n(f)\|/E_n(f)$ BOUNDED?

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(Communicated by J. Marshall Ash)

Dedicated to Professor Tingfan Xie for his sixtieth birthday

ABSTRACT. Let $\{\phi_n\}_{n=1}^{\infty}$ be a positive increasing sequence and $\phi_n \hat{f}(n)$ decrease. We ask what exact conditions on $\{\phi_n\}$ make $\|f - S_n(f)\|/E_n(f)$ bounded? The present paper will give a complete answer to it.

1. INTRODUCTION

Let $C_{2\pi}$ be the space of all real continuous functions of period 2π with norm $\|\cdot\| = \max_{-\infty<x<\infty} |\cdot|$. For an even function $f \in C_{2\pi}$ (we always assume that $f(x)$ is an even function), denote its Fourier series by

$$f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n) \cos nx,$$

and the nth partial sum of the Fourier series by

$$S_n(f, x) = \sum_{k=0}^{n} \hat{f}(k) \cos kx.$$

In Fourier analysis, it is always an interesting question whether or not the approximation by partial sums of a continuous function can achieve the best approximation rate in uniform norm. It is well known that (cf. [Zyg])

$$\|f - S_n(f)\| = O(\log(n+1)E_n(f)), \tag{1}$$

where $E_n(f)$ is the best approximation by trigonometric polynomials of degree n. We also know that (cf. [Zyg]), in general, the factor $\log(n+1)$ in (1) cannot be removed. However, people always hope that, for $f(x)$ in some subclass of $C_{2\pi}$ (especially in those which can be easily characterized like continuous functions with restricted Fourier coefficients), one can obtain some better estimates, preferably,

$$\|f - S_n(f)\| = O(E_n(f)). \tag{2}$$

Received by the editors June 22, 1992 and, in revised form, October 10, 1992.
1991 *Mathematics Subject Classification*. Primary 42A10, 42A20, 42A32.
Key words and phrases. Approximation, Fourier sums, coefficients, strong monotonicity.
Supported in part by NSERC Canada.

One result in this direction is due to [NeRi] showing that

$$\|f - S_n(f)\| \le 4eE_n(f)$$

holds for every function $f \in C_{2\pi}$ with decreasing and logarithmically concave Fourier coefficients (that a sequence a_n is logarithmically concave means that it satisfies $a_n^2 \ge a_{n+1}a_{n-1}$ for every n).

At the same time, we can easily see that for $f \in C_{2\pi}$ with positive Fourier coefficients, consequently for $f \in C_{2\pi}$ with monotone Fourier coefficients,

$$\|f - S_n(f)\| = O(\omega_k(f, n^{-1})), \qquad k \ge 1,$$

obviously holds, where $\omega_k(f, t)$ is the modulus of smoothness of order k of $f(x)$, and the O in the above inequality depends upon k only (the same result also holds for odd continuous functions of period 2π by a little more complicated arguments). Probably from this point of view, [Xie] asked

Problem X1. Does there exist a positive constant M such that, for every function $f \in C_{2\pi}$ with positive Fourier coefficients and all $n = 1, 2, \dots,$

$$\|f - S_n(f)\| \le ME_n(f)?$$

Later, in a seminar, Xie asked further

Problem X2. Does there exist a positive constant M such that, for every function $f \in C_{2\pi}$ with decreasing Fourier coefficients and all $n = 1, 2, \dots,$

$$\|f - S_n(f)\| \le ME_n(f)?$$

A little surprisingly, [Zho] constructed a counterexample showing that there exists a function $f \in C_{2\pi}$ with strongly monotone Fourier coefficients (which means that $n\hat{f}(n)$ decreases, in symbol, $n\hat{f}(n) \downarrow$) such that

$$\limsup_{n\to\infty} \frac{\|f - S_n(f)\|}{\log n E_n(f)} > 0.$$

This gave a negative answer to Problem X2 as well as to Problem X1.

Some natural questions now arise: What kinds of strong monotonicity conditions on Fourier coefficients can guarantee (2)? Is there any counterexample which satisfies $n^s\hat{f}(n) \downarrow$ for sufficiently large $s > 0$ (the counterexample given in [Zho] does not satisfy this condition) but makes $\|f - S_n(f)\|/E_n(f)$ unbounded?

One can easily verify a trivial example that if $q^n\hat{f}(n) \downarrow$ for some $q > 1$, then (2) holds. Therefore, a general statement of the problem must be as follows. Let $\Phi = \{\phi_n\}_{n=1}^{\infty}$ be a positive increasing sequence and $\phi_n\hat{f}(n) \downarrow$. Then what are exact conditions on $\{\phi_n\}$ which make $\|f - S_n(f)\|/E_n(f)$ bounded?

The present paper will give a complete answer to this question. We always let C indicate some positive absolute constant which is not necessarily the same in every different occurrence.

2. Result and proof

Given $\varepsilon > 0$. Assume $\Phi = \{\phi_n\}_{n=1}^{\infty}$ is a positive increasing sequence. Write $L_n(\Phi, \varepsilon)$ to be the maximum length of string of $\{\phi_{k+1}/\phi_k\}_{k=1}^{n-1}$ not exceeding

$1+\varepsilon$, namely,

$$L_n(\Phi, \varepsilon) = \max_{1\le j<k\le n-1} \{k-j+1 : \phi_{j+1}/\phi_j \le 1+\varepsilon, \\ \phi_{j+2}/\phi_{j+1} \le 1+\varepsilon, \dots, \phi_{k+1}/\phi_k \le 1+\varepsilon\},$$

in case there is no $\phi_{j+1}/\phi_j \le 1+\varepsilon$ for $1 \le j \le n-1$, we simply set $L_n(\Phi, \varepsilon) = 0$. Since $L_{n+1}(\Phi, \varepsilon) \ge L_n(\Phi, \varepsilon)$, we may define the limit

$$L(\Phi, \varepsilon) = \lim_{n\to\infty} L_n(\Phi, \varepsilon).$$

Define

$$\mathscr{M}_\phi = \left\{ f \in C_{2\pi} : f \sim \sum_{n=0}^{\infty} \hat{f}(n) \cos nx, \ \phi_n \hat{f}(n) \downarrow \right\}$$

and

$$\mathscr{S}_E = \{ f \in C_{2\pi} : \|f - S_n(f)\| = O(E_n(f)) \}.$$

We have the following result.

Theorem 1. *The necessary and sufficient condition for $\mathscr{M}_\phi \subset \mathscr{S}_E$ is $L(\Phi, \varepsilon_0) = O(1)$ for some $\varepsilon_0 > 0$.*

Lemma 1. *$L(\Phi, \varepsilon) = \infty$ for all $\varepsilon > 0$ implies that there are two increasing sequences of natural numbers $\{n_l\}_{l=1}^\infty$ and $\{m_l\}_{l=1}^\infty$ with $n_l + m_l + 1 < n_{l+1}$, $l = 1, 2, \dots$, as well as a decreasing sequence $\{\delta_l\}_{l=1}^\infty$ with $\lim_{l\to\infty} \delta_l = 0$ such that*

$$\phi_{n_l+j+1}/\phi_{n_l+j} \le 1+\delta_l, \qquad j = 0, 1, \dots, m_l, \ l = 1, 2, \dots. \tag{3}$$

Proof. The argument is elementary and we omit it. □

Lemma 2. *Let $L(\Phi, \varepsilon) = \infty$ for all $\varepsilon > 0$. Then there is a function $f \in \mathscr{M}_\Phi$ such that*

$$\limsup_{n\to\infty} \frac{\|f - S_n(f)\|}{E_n(f)} = \infty.$$

Proof. Since ϕ_n is positive and increasing, there is a constant $c > 0$ such that $\phi_n \ge c$ for all $n = 1, 2, \dots$. Without loss of generality we therefore assume $\phi_n \ge 1$ for all $n = 1, 2, \dots$.[1] From Lemma 1, we have three sequences $\{n_l\}$, $\{m_l\}$, and $\{\delta_l\}$ making (3) be satisfied. Then we may also assume $\delta_l < 1/e^2$ and $3 < m_l < n_l$ for $l = 1, 2, \dots$. We construct the required counterexample by induction. Let $\hat{f}(0)$ be any positive number and $\hat{f}(1) = 1$. Given $\hat{f}(k)$, we have the following two cases.

Case 1. $k \ne n_l, n_l+1, \dots, n_l+m_l$, $l = 1, 2, \dots$. In this case, we simply set

$$\hat{f}(k+1) = \hat{f}(k)/(2\phi_{k+1}), \tag{4}$$

in particular,

$$\hat{f}(n_l) = \hat{f}(n_l - 1)/(2\phi_{n_l}).$$

Case 2. $k = n_l$ *for some* l. Now we construct

$$\hat{f}(n_l + j) = \frac{\hat{f}(n_l)}{2j\phi_{n_l+1}\log(M_l+1)}, \tag{5}$$

$$j = 1, 2, \dots, \ M_l := \min\left\{ m_l, \left[\sqrt{\delta_l^{-1}}\right] \right\} < e,$$

[1]Otherwise write $\phi_n^* = \phi_n/c$. Then $\phi_n^* \ge 1$ and we use ϕ_n^* instead.

where $[x]$ is the greatest integer not exceeding x,

$$\hat{f}(n_l+j+1)=\frac{\hat{f}(n_l+j)}{2\phi_{n_l+j+1}\log(M_l+1)}, \qquad j=M_l,\ M_l+1,\dots,m_l. \tag{6}$$

In this way we have defined a function

$$f(x)=\sum_{n=0}^{\infty}\hat{f}(n)\cos nx.$$

Clearly, by condition (5), for $l=1,2,\dots,$

$$\sum_{k=n_l+1}^{n_l+M_l}\hat{f}(k)\le\frac{\hat{f}(n_l)}{2\phi_{n_l+1}}\le\frac{\hat{f}(n_l)}{2}, \tag{7}$$

while by conditions (4) and (6), for $k\ne n_l+1,\ n_l+2,\dots,\ n_l+M_l-1$,

$$\hat{f}(k+1)\le\tfrac{1}{2}\hat{f}(k). \tag{8}$$

Altogether, $\sum_{n=0}^{\infty}\hat{f}(n)<\infty$, which means $f\in C_{2\pi}$. We check that

$$\begin{aligned}
f(0)-S_{n_l}(f,0)&=\sum_{k=n_l+1}^{n_l+M_l}\hat{f}(k)+\sum_{k=n_l+M_l+1}^{\infty}\hat{f}(k)\\
&\ge\frac{\hat{f}(n_l)}{\phi_{n_l+1}\log(M_l+1)}\log M_l-O\left(\frac{\hat{f}(n_l+M_l)}{2\phi_{n_l+M_l+1}\log(M_l+1)}\right)\qquad\text{(by (7), (8))}\\
&\ge C\frac{\hat{f}(n_l)}{\phi_{n_l+1}}.
\end{aligned}$$

At the same time,

$$\begin{aligned}
E_{n_l}(f)\le&\left\|\sum_{j=1}^{M_l}\hat{f}(n_l+j)\cos(n_l+j)x-\sum_{j=1}^{M_l}\hat{f}(n_l+j)\cos(n_l-j)x\right\|\\
&+\sum_{k=n_l+M_l+1}^{\infty}\hat{f}(k).
\end{aligned}$$

By noting that

$$\begin{aligned}
&\sum_{j=1}^{M_l}\hat{f}(n_l+j)\cos(n_l+j)x-\sum_{j=1}^{M_l}\hat{f}(n_l+j)\cos(n_l-j)x\\
&\qquad=-\frac{\hat{f}(n_l)\sin n_lx}{\phi_{n_l+1}\log(M_l+1)}\sum_{j=1}^{M_l}\frac{\sin jx}{j},
\end{aligned}$$

together with the well-known inequality (cf. [Zyg])

$$\sup_{m\ge1}\left\|\sum_{k=1}^{m}\frac{\sin kx}{k}\right\|\le3\sqrt{\pi},$$

we have

$$E_{n_l}(f) \le \frac{3\sqrt{\pi}\hat{f}(n_l)}{\phi_{n_l+1}\log(M_l+1)} + O\left(\frac{\hat{f}(n_l+M_l)}{2\phi_{n_l+M_l+1}\log(M_l+1)}\right)$$
$$= O\left(\frac{\hat{f}(n_l)}{\phi_{n_l+1}\log(M_l+1)}\right).$$

Therefore,

$$\frac{\|f - S_{n_l}(f)\|}{E_{n_l}(f)} \ge C\log M_l.$$

Since $\lim_{l\to\infty} M_l = \infty$ by $\lim_{l\to\infty}\delta_l = 0$ and $\lim_{l\to\infty} m_l = \infty$, the above estimate yields

$$\lim_{l\to\infty}\frac{\|f - S_{n_l}(f)\|}{E_{n_l}(f)} = \infty.$$

We now need to verify that $f \in \mathscr{M}_\Phi$. If $k \ne n_l+1,\ n_l+2, \dots, n_l+M_l-1$, $l = 1, 2, \dots$, by (4), (6),

$$\phi_{k+1}\hat{f}(k+1) \le \frac{\hat{f}(k)}{2} < \phi_k\hat{f}(k).$$

When $k = n_l + j$, $j = 1, 2, \dots, M_l - 1$, for some l,

$$\phi_{k+1}\hat{f}(k+1) - \phi_k\hat{f}(k) = \frac{\hat{f}(n_l)}{2\phi_{n_l+1}\log(M_l+1)}\left(\frac{\phi_{n_l+j+1}}{j+1} - \frac{\phi_{n_l+j}}{j}\right)$$
$$= \frac{\hat{f}(n_l)}{2\phi_{n_l+1}\log(M_l+1)}\frac{\phi_{n_l+j}}{j+1}\left(\frac{\phi_{n_l+j+1}}{\phi_{n_l+j}} - 1 - \frac{1}{j}\right).$$

By Lemma 1,

$$\phi_{n_l+j+1}/\phi_{n_l+j} \le 1 + \delta_l,$$

but for $1 \le j \le M_l - 1$,

$$1/j \ge 1/(M_l - 1) \ge \sqrt{\delta_l},$$

so that

$$\phi_{n_l+j+1}/\phi_{n_l+j} - 1 - 1/j \le \delta_l - \sqrt{\delta_l} < 0.$$

Altogether it follows that for $k = n_l + j$, $j = 1, 2, \dots, M_l - 1$,

$$\phi_{k+1}\hat{f}(k+1) - \phi_k\hat{f}(k) < 0.$$

Thus we are done. □

Lemma 3. *Assume* $L(\Phi, \varepsilon_0) = O(1)$ *for some* $\varepsilon_0 > 0$. *Let* $f \in \mathscr{M}_\Phi$. *Then for* $n \ge 1$,

$$\sum_{k=n+1}^{\infty}\hat{f}(k) = O(\hat{f}(n+1)).$$

Proof. Given $n \ge 1$. Since $L(\Phi, \varepsilon_0) = O(1)$, say, $L(\Phi, \varepsilon_0) \le M$, we have a sequence $\{n_l\} \subset \{n+1, n+2, \dots\}$ such that $0 \le n_l - n - 1 \le M$, $0 \le n_{l+1} - n_l \le M$, and

$$\phi_{n_l+1}/\phi_{n_l} \ge 1 + \varepsilon_0. \tag{9}$$

Write

$$\sum_{k=n+1}^{\infty} \hat{f}(k) = \left(\sum_{k=n+1}^{n_l} + \sum_{l=1}^{\infty} \sum_{k=n_l+1}^{n_{l+1}} \right) \hat{f}(k).$$

Evidently, for $k \in \{n_l + 2, n_l + 3, \dots, n_{l+1}\}$,

$$\hat{f}(k) = \phi_k \hat{f}(k) \frac{1}{\phi_k} \le \phi_{n_l+1} \hat{f}(n_l + 1) \frac{1}{\phi_k} \le \hat{f}(n_l + 1),$$

and similarly for $k = n+1, n+2, \dots, n_l$, $\hat{f}(k) \le \hat{f}(n+1)$. Hence

$$\begin{aligned}
\sum_{k=n+1}^{\infty} \hat{f}(k) &\le M \left(\hat{f}(n+1) + \sum_{l=1}^{\infty} \hat{f}(n_l + 1) \right) \\
&\le M \left(\hat{f}(n+1) + \phi_{n_1+1} \hat{f}(n_1 + 1) \sum_{l=1}^{\infty} \frac{1}{\phi_{n_l+1}} \right) \\
&= M \left(\hat{f}(n+1) + \hat{f}(n_1 + 1) \sum_{l=1}^{\infty} \frac{\phi_{n_1+1}}{\phi_{n_l+1}} \right) \\
&\le M \hat{f}(n+1) \left(1 + \sum_{l=0}^{\infty} (1 + \varepsilon_0)^{-l} \right) \quad \text{(by (9))} \\
&= O(\hat{f}(n+1)). \quad \square
\end{aligned}$$

Proof of Theorem 1. *Necessity.* By Lemma 2, we have the required result.

Sufficiency. From Lemma 3,

$$\|f - S_n(f)\| = \sum_{k=n+1}^{\infty} \hat{f}(k) = O(\hat{f}(n+1)).$$

On the other hand,

$$\begin{aligned}
E_n(f) &\ge \left(\int_0^{2\pi} |f(x) - S_n(f, x)|^2 \, dx \right)^{1/2} \\
&= \left(\sum_{k=n+1}^{\infty} \hat{f}^2(k) \right)^{1/2} \ge \hat{f}(n+1).
\end{aligned}$$

That is,

$$\|f - S_n(f)\| = O(E_n(f)),$$

or $\mathscr{M}_\Phi \subset \mathscr{S}_E$. $\square$

The following are two typical applications of Theorem 1.

Corollary 1. *Let $s > 0$ be any given number. Then there exists a function $f \in C_{2\pi}$ with $n^s \hat{f}(n) \downarrow$ such that*

$$\limsup_{n \to \infty} \frac{\|f - S_n(f)\|}{E_n(f)} = \infty.$$

Corollary 2. *If $f \in C_{2\pi}$ satisfies $q^n \hat{f}(n) \downarrow$ for some $q > 1$, then*

$$\|f - S_n(f)\| = O(E_n(f)).$$

For sine series, we have a similar result (with a similar proof).

Theorem 2. *Let* $g(x) \in C_{2\pi}$,

$$g(x) \sim \sum_{n=1}^{\infty} \hat{g}(n) \sin nx.$$

Then the necessary and sufficient condition for $\mathscr{M}_\phi^* \subset \mathscr{S}_E$ *is* $L(\Phi, \varepsilon_0) = O(1)$ *for some* $\varepsilon_0 > 0$, *where*

$$\mathscr{M}_\phi^* = \left\{ g \in C_{2\pi} \colon g \sim \sum_{n=1}^{\infty} \hat{g}(n) \sin nx, \ \phi_n \hat{g}(n) \downarrow \right\}.$$

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