WHAT STRONG MONOTONICITY CONDITION ON FOURIER COEFFICIENTS CAN MAKE THE RATIO \\
\|f - S_n(f)\|/E_n(f) \text{ BOUNDED?} \\

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Dedicated to Professor Tingfan Xie for his sixtieth birthday \\

Abstract. Let \( \{\phi_n\}_{n=1}^{\infty} \) be a positive increasing sequence and \( \phi_n \hat{f}(n) \) decrease. We ask what exact conditions on \( \{\phi_n\} \) make \( \|f - S_n(f)\|/E_n(f) \) bounded? The present paper will give a complete answer to it. 

1. Introduction 

Let \( C_{2\pi} \) be the space of all real continuous functions of period \( 2\pi \) with norm \\
\[ \| \cdot \| = \max_{-\infty < x < \infty} | \cdot |. \] 
For an even function \( f \in C_{2\pi} \) (we always assume that \( f(x) \) is an even function), denote its Fourier series by 

\[ f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n) \cos nx, \]

and the \( n \)th partial sum of the Fourier series by 

\[ S_n(f, x) = \sum_{k=0}^{n} \hat{f}(k) \cos kx. \]

In Fourier analysis, it is always an interesting question whether or not the approximation by partial sums of a continuous function can achieve the best approximation rate in uniform norm. It is well known that (cf. [Zyg]) 

\[ \|f - S_n(f)\| = O(\log(n + 1)E_n(f)), \]

where \( E_n(f) \) is the best approximation by trigonometric polynomials of degree \( n \). We also know that (cf. [Zyg]), in general, the factor \( \log(n + 1) \) in (1) cannot be removed. However, people always hope that, for \( f(x) \) in some subclass of \( C_{2\pi} \) (especially in those which can be easily characterized like continuous functions with restricted Fourier coefficients), one can obtain some better estimates, preferably, 

\[ \|f - S_n(f)\| = O(E_n(f)). \]

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One result in this direction is due to [NeRi] showing that
\[ \|f - S_n(f)\| \leq 4E_n(f) \]
holds for every function \( f \in C_{2\pi} \) with decreasing and logarithmically concave Fourier coefficients (that a sequence \( a_n \) is logarithmically concave means that it satisfies \( a_n^2 \geq a_{n+1}a_{n-1} \) for every \( n \)).

At the same time, we can easily see that for \( f \in C_{2\pi} \) with positive Fourier coefficients, consequently for \( f \in C_{2\pi} \) with monotone Fourier coefficients,
\[ \|f - S_n(f)\| = O(\omega_k(f, n^{-1})), \quad k \geq 1, \]
obviously holds, where \( \omega_k(f, t) \) is the modulus of smoothness of order \( k \) of \( f(x) \), and the \( O \) in the above inequality depends upon \( k \) only (the same result also holds for odd continuous functions of period \( 2\pi \) by a little more complicated arguments). Probably from this point of view, [Xie] asked

**Problem X1.** Does there exist a positive constant \( M \) such that, for every function \( f \in C_{2\pi} \) with positive Fourier coefficients and all \( n = 1, 2, \ldots, \)
\[ \|f - S_n(f)\| \leq ME_n(f)? \]

Later, in a seminar, Xie asked further

**Problem X2.** Does there exist a positive constant \( M \) such that, for every function \( f \in C_{2\pi} \) with decreasing Fourier coefficients and all \( n = 1, 2, \ldots, \)
\[ \|f - S_n(f)\| \leq ME_n(f)? \]

A little surprisingly, [Zho] constructed a counterexample showing that there exists a function \( f \in C_{2\pi} \) with strongly monotone Fourier coefficients (which means that \( n\hat{f}(n) \) decreases, in symbol, \( n\hat{f}(n) \downarrow \)) such that
\[ \limsup_{n \to \infty} \frac{\|f - S_n(f)\|}{\log n E_n(f)} > 0. \]

This gave a negative answer to Problem X2 as well as to Problem X1.

Some natural questions now arise: What kinds of strong monotonicity conditions on Fourier coefficients can guarantee (2)? Is there any counterexample which satisfies \( n^s \hat{f}(n) \downarrow \) for sufficiently large \( s > 0 \) (the counterexample given in [Zho] does not satisfy this condition) but makes \( \|f - S_n(f)\|/E_n(f) \) unbounded?

One can easily verify a trivial example that if \( q^n \hat{f}(n) \downarrow \) for some \( q > 1 \), then (2) holds. Therefore, a general statement of the problem must be as follows.

Let \( \Phi = \{\phi_n\}_{n=1}^{\infty} \) be a positive increasing sequence and \( \phi_n \hat{f}(n) \downarrow \). Then what are exact conditions on \( \{\phi_n\} \) which make \( \|f - S_n(f)\|/E_n(f) \) bounded?

The present paper will give a complete answer to this question. We always let \( C \) indicate some positive absolute constant which is not necessarily the same in every different occurrence.

**2. Result and proof**

Given \( \varepsilon > 0 \). Assume \( \Phi = \{\phi_n\}_{n=1}^{\infty} \) is a positive increasing sequence. Write \( L_n(\Phi, \varepsilon) \) to be the maximum length of string of \( \{\phi_{k+1}/\phi_k\}_{k=1}^{n-1} \) not exceeding
$1 + \varepsilon$, namely,
\[
L_n(\Phi, \varepsilon) = \max_{1 \leq j < k \leq n-1} \{ k - j + 1 : \phi_j+1/\phi_j \leq 1 + \varepsilon, \phi_{j+2}/\phi_{j+1} \leq 1 + \varepsilon, \ldots, \phi_k+1/\phi_k \leq 1 + \varepsilon \},
\]
in case there is no $\phi_j+1/\phi_j \leq 1 + \varepsilon$ for $1 \leq j \leq n-1$, we simply set $L_n(\Phi, \varepsilon) = 0$. Since $L_{n+1}(\Phi, \varepsilon) \geq L_n(\Phi, \varepsilon)$, we may define the limit
\[
L(\Phi, \varepsilon) = \lim_{n \to \infty} L_n(\Phi, \varepsilon).
\]

Define
\[
\mathcal{M}_\Phi = \left \{ f \in C_2 : f \sim \sum_{n=0}^{\infty} \hat{f}(n) \cos nx, \phi_n \hat{f}(n) \downarrow \right \}
\]
and
\[
\mathcal{E}_E = \{ f \in C_2 : \|f - S_n(f)\| = O(E_n(f)) \}.
\]
We have the following result.

**Theorem 1.** The necessary and sufficient condition for $\mathcal{M}_\Phi \subset \mathcal{E}_E$ is $L(\Phi, \varepsilon_0) = O(1)$ for some $\varepsilon_0 > 0$.

**Lemma 1.** $L(\Phi, \varepsilon) = \infty$ for all $\varepsilon > 0$ implies that there are two increasing sequences of natural numbers $\{n_l\}_{l=1}^{\infty}$ and $\{m_l\}_{l=1}^{\infty}$ with $n_l + m_l + 1 < n_{l+1}$, $l = 1, 2, \ldots$, as well as a decreasing sequence $\{\delta_l\}_{l=1}^{\infty}$ with $\lim_{l \to \infty} \delta_l = 0$ such that
\[
\phi_{n_l+j+1}/\phi_{n_l+j} \leq 1 + \delta_l, \quad j = 0, 1, \ldots, m_l, \quad l = 1, 2, \ldots.
\]

**Proof.** The argument is elementary and we omit it. □

**Lemma 2.** Let $L(\Phi, \varepsilon) = \infty$ for all $\varepsilon > 0$. Then there is a function $f \in M_\Phi$ such that
\[
\limsup_{n \to \infty} \frac{\|f - S_n(f)\|}{E_n(f)} = \infty.
\]

**Proof.** Since $\phi_n$ is positive and increasing, there is a constant $c > 0$ such that $\phi_n \geq c$ for all $n = 1, 2, \ldots$. Without loss of generality we therefore assume $\phi_n \geq 1$ for all $n = 1, 2, \ldots$. From Lemma 1, we have three sequences $\{n_l\}, \{m_l\}$, and $\{\delta_l\}$ making (3) be satisfied. Then we may also assume $\delta_l < 1/e^2$ and $3 < m_l < n_l$ for $l = 1, 2, \ldots$. We construct the required counterexample by induction. Let $\hat{f}(0)$ be any positive number and $\hat{f}(1) = 1$. Given $\hat{f}(k)$, we have the following two cases.

**Case 1.** $k \neq n_l, n_l + 1, \ldots, n_l + m_l, \quad l = 1, 2, \ldots$. In this case, we simply set
\[
\hat{f}(k+1) = \hat{f}(k)/(2\phi_{k+1}),
\]
in particular,
\[
\hat{f}(n_l) = \hat{f}(n_l-1)/(2\phi_{n_l}).
\]

**Case 2.** $k = n_l$ for some $l$. Now we construct
\[
\hat{f}(n_l+j) = \frac{\hat{f}(n_l)}{2j\phi_{n_l+1} \log(M_l + 1)},
\]
with
\[
j = 1, 2, \ldots, M_l := \min \left \{ m_l, \left \lfloor \sqrt{\delta_l^{-1}} \right \rfloor \right \} < e.
\]

\[\text{otherwise write } \phi_n^* = \phi_n/c. \text{ Then } \phi_n^* \geq 1 \text{ and we use } \phi_n^* \text{ instead.}]

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where \([x]\) is the greatest integer not exceeding \(x\),

\[
(6) \quad \hat{f}(n_l + j + 1) = \frac{\hat{f}(n_l + j)}{2\phi n_l + j + 1 \log(M_l + 1)}, \quad j = M_l, M_l + 1, \ldots, m_l.
\]

In this way we have defined a function

\[
f(x) = \sum_{n=0}^{\infty} \hat{f}(n) \cos nx.
\]

Clearly, by condition (5), for \(l = 1, 2, \ldots\),

\[
(7) \quad \frac{n_l + M_l}{k = n_l + 1} \hat{f}(k) \leq \frac{\hat{f}(n_l)}{2\phi n_l + 1} \leq \frac{\hat{f}(n_l)}{2},
\]

while by conditions (4) and (6), for \(k \neq n_l + 1, n_l + 2, \ldots, n_l + M_l - 1\),

\[
(8) \quad \hat{f}(k + 1) \leq \frac{1}{2} \hat{f}(k).
\]

Altogether, \(\sum_{n=0}^{\infty} \hat{f}(n) < \infty\), which means \(f \in C_{2\pi}\). We check that

\[
f(0) - S_{n_l}(f, 0) = \sum_{k = n_l + 1}^{n_l + M_l} \hat{f}(k) + \sum_{k = n_l + M_l + 1}^{\infty} \hat{f}(k)
\]

\[
\geq \frac{\hat{f}(n_l)}{\phi n_l + 1 \log(M_l + 1)} \log M_l - O \left( \frac{\hat{f}(n_l + M_l)}{2\phi n_l + M_l + 1 \log(M_l + 1)} \right) \quad \text{(by (7), (8))}
\]

At the same time,

\[
E_{n_l}(f) \leq \left\| \sum_{j=1}^{M_l} \hat{f}(n_l + j) \cos(n_l + j)x - \sum_{j=1}^{M_l} \hat{f}(n_l + j) \cos(n_l - j)x \right\|
\]

\[+ \sum_{k = n_l + M_l + 1}^{\infty} \hat{f}(k).
\]

By noting that

\[
\sum_{j=1}^{M_l} \hat{f}(n_l + j) \cos(n_l + j)x - \sum_{j=1}^{M_l} \hat{f}(n_l + j) \cos(n_l - j)x
\]

\[= - \frac{\hat{f}(n_l)}{\phi n_l + 1 \log(M_l + 1)} \sum_{j=1}^{M_l} \sin jx \frac{\sin jx}{j},
\]

together with the well-known inequality (cf. [Zyg])

\[
\sup_{m \geq 1} \left\| \sum_{k=1}^{m} \frac{\sin kx}{k} \right\| \leq 3\sqrt{\pi},
\]
we have

\[
E_n(f) \leq \frac{3\sqrt{\pi} \hat{f}(n_l)}{\phi_{n+1} \log(M_l + 1)} + O\left(\frac{\hat{f}(n_l + M_l)}{2\phi_{n+1} \log(M_l + 1)} \right)
\]

Therefore,

\[
\frac{\|f - S_n(f)\|}{E_n(f)} \geq C \log M_l.
\]

Since \(\lim_{l \to \infty} M_l = \infty\) by \(\lim_{l \to \infty} \delta_l = 0\) and \(\lim_{l \to \infty} m_l = \infty\), the above estimate yields

\[
\lim_{l \to \infty} \frac{\|f - S_n(f)\|}{E_n(f)} = \infty.
\]

We now need to verify that \(f \in \mathcal{M}\). If \(k \neq n_l + 1, n_l + 2, \ldots, n_l + M_l - 1, l = 1, 2, \ldots\), by (4), (6),

\[
\phi_{k+1} \hat{f}(k + 1) \leq \frac{\hat{f}(k)}{2} < \phi_k \hat{f}(k).
\]

When \(k = n_l + j, j = 1, 2, \ldots, M_l - 1,\) for some \(l,\)

\[
\phi_{k+1} \hat{f}(k + 1) - \phi_k \hat{f}(k) = \frac{\hat{f}(n_l)}{2\phi_{n+1} \log(M_l + 1)} \left( \frac{\phi_{n_l+j+1}}{j+1} - \frac{\phi_{n_l+j}}{j} \right)
\]

\[
= \frac{\hat{f}(n_l)}{2\phi_{n+1} \log(M_l + 1)} \phi_{n_l+j} \left( \frac{\phi_{n_l+j+1}}{\phi_{n_l+j}} - 1 - \frac{1}{j} \right).
\]

By Lemma 1,

\[
\phi_{n_l+j+1}/\phi_{n_l+j} \leq 1 + \delta_l,
\]

but for \(1 \leq j \leq M_l - 1,\)

\[
1/j \geq 1/(M_l - 1) \geq \sqrt{\delta_l},
\]

so that

\[
\phi_{n_l+j+1}/\phi_{n_l+j} - 1 - 1/j \leq \delta_l - \sqrt{\delta_l} < 0.
\]

Altogether it follows that for \(k = n_l + j, j = 1, 2, \ldots, M_l - 1,\)

\[
\phi_{k+1} \hat{f}(k + 1) - \phi_k \hat{f}(k) < 0.
\]

Thus we are done. \(\Box\)

**Lemma 3.** Assume \(L(\Phi, \epsilon_0) = O(1)\) for some \(\epsilon_0 > 0\). Let \(f \in \mathcal{M}\). Then for \(n \geq 1,\)

\[
\sum_{k=n+1}^{\infty} \hat{f}(k) = O(\hat{f}(n + 1)).
\]

**Proof.** Given \(n \geq 1.\) Since \(L(\Phi, \epsilon_0) = O(1),\) say, \(L(\Phi, \epsilon_0) \leq M,\) we have a sequence \(\{n_l\} \subseteq \{n + 1, n + 2, \ldots\}\) such that \(0 \leq n_l - n - 1 \leq M, 0 \leq n_{l+1} - n_l \leq M,\) and

\[
\phi_{n_l+1}/\phi_{n_l} \geq 1 + \epsilon_0.
\]
Write
\[ \sum_{k=n+1}^\infty \hat{f}(k) = \left( \sum_{k=n+1}^{n_1} + \sum_{l=1}^\infty \sum_{k=n_l+1}^{n_{l+1}} \right) \hat{f}(k). \]

Evidently, for \( k \in \{n_l + 2, n_l + 3, \ldots, n_{l+1}\} \),
\[ \hat{f}(k) = \phi_k \hat{f}(k) \frac{1}{\phi_k} \leq \phi_{n_l+1} \hat{f}(n_l + 1) \frac{1}{\phi_k} \leq \hat{f}(n_l + 1), \]
and similarly for \( k = n + 1, n + 2, \ldots, n_l, \hat{f}(k) \leq \hat{f}(n + 1) \). Hence
\[ \sum_{k=n+1}^\infty \hat{f}(k) \leq M \left( \hat{f}(n + 1) + \sum_{l=1}^\infty \hat{f}(n_l + 1) \right) \]
\[ \leq M \left( \hat{f}(n + 1) + \phi_{n_l+1} \hat{f}(n_l + 1) \sum_{l=1}^\infty \frac{1}{\phi_{n_l+1}} \right) \]
\[ = M \left( \hat{f}(n + 1) + \frac{\hat{f}(n_l + 1)}{\phi_{n_l+1}} \sum_{l=1}^\infty \frac{\phi_{n_l+1}}{\phi_{n_l+1}} \right) \]
\[ \leq M \hat{f}(n + 1) \left( 1 + \sum_{l=0}^\infty (1 + \varepsilon_0)^{-l} \right) \quad \text{(by (9))} \]
\[ = O(\hat{f}(n + 1)). \quad \Box \]

**Proof of Theorem 1. Necessity.** By Lemma 2, we have the required result.

**Sufficiency.** From Lemma 3,
\[ \|f - S_n(f)\| = \sum_{k=n+1}^\infty \hat{f}(k) = O(\hat{f}(n + 1)). \]
On the other hand,
\[ E_n(f) \geq \left( \int_0^{2\pi} |f(x) - S_n(f, x)|^2 \, dx \right)^{1/2} \]
\[ = \left( \sum_{k=n+1}^\infty \hat{f}^2(k) \right)^{1/2} \geq \hat{f}(n + 1). \]
That is,
\[ \|f - S_n(f)\| = O(E_n(f)), \]
or \( M \Phi \subseteq \mathcal{F}_E \). \( \Box \)

The following are two typical applications of Theorem 1.

**Corollary 1.** Let \( s > 0 \) be any given number. Then there exists a function \( f \in C_{2\pi} \) with \( n^s \hat{f}(n) \downarrow \) such that
\[ \lim_{n \to \infty} \sup \frac{\|f - S_n(f)\|}{E_n(f)} = \infty. \]

**Corollary 2.** If \( f \in C_{2\pi} \) satisfies \( q^nf(n) \downarrow \) for some \( q > 1 \), then
\[ \|f - S_n(f)\| = O(E_n(f)). \]
For sine series, we have a similar result (with a similar proof).
Theorem 2. Let \( g(x) \in C_{2\pi} \),
\[
g(x) \sim \sum_{n=1}^{\infty} \hat{g}(n) \sin nx.
\]
Then the necessary and sufficient condition for \( \mathcal{M}_\phi^* \subset \mathcal{S}_E \) is \( L(\Phi, \varepsilon_0) = O(1) \) for some \( \varepsilon_0 > 0 \), where
\[
\mathcal{M}_\phi^* = \left\{ g \in C_{2\pi} : g \sim \sum_{n=1}^{\infty} \hat{g}(n) \sin nx, \quad \phi_n \hat{g}(n) \downarrow \right\}.
\]

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