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CENTRAL SEQUENCES IN SUBFACTORS II

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ABSTRACT. We studied in *On the existence of central sequences in subfactors* [Trans. Amer. Math. Soc. **321** (1990), 117–128] certain ergodicity properties of inclusions of II_1 factors $N \subset M$. We give here various explicit examples of pairs of II_1 factors $N \subset M$ which have or do not have these properties. In particular, we show that if $N \subset M$ are hyperfinite II_1 factors with finite Jones's index, then both situations may occur.

1. INTRODUCTION

We studied in [Bi1] certain asymptotic commutativity properties of inclusions of II_1 factors $N \subset M$. Namely, we gave a list of equivalent conditions on the pair $N \subset M$ that assure that the subfactor N contains nontrivial central sequences for the ambient algebra M. Let us recall the definition:

Definition 1.1. Let $N \subset M$ be (separable) II_1 factors with faithful normal normalized trace τ. We say that $N \subset M$ has *property* Γ if given any $\epsilon > 0$ and any operators $x_1, \dots, x_n \in M$, there is a unitary $u \in N$, $\tau(u) = 0$, such that $\|[x_i, u]\|_2 \leq \epsilon$, $1 \leq i \leq n$.

The study of central sequences of M contained in the subfactor N is closely related to deciding whether a given subfactor of the hyperfinite II_1 factor is *amenable* [Po3, Po7], i.e., classified by its higher relative commutant invariant, or not. The existence of nontrivial central sequences for M contained in the subfactor N is a necessary condition for the amenability of N in the sense of Popa [Po7]. However, as we shall see below, there are many examples of non-amenable subfactors of the hyperfinite II_1 factor R that do contain nontrivial central sequences for the ambient factor.

If ω is a free ultrafilter of $\mathbb{N}$, then $N \subset M$ has property Γ if and only if $M' \cap N^\omega \neq \mathbb{C}$. We proved in [Bi1] that in this case the algebra $M' \cap N^\omega$ contains a diffuse abelian subalgebra.

In this note we will give examples of inclusions of II_1 factors $N \subset M$ such that N and M have both property Γ in the sense of Murray and von Neumann ([MvN]), however, the inclusion $N \subset M$ does not have property Γ (Definition 1.1). In particular, we will show that if N and M are the hyperfinite II_1 factor

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with finite Jones's index $[M : N]$ ([Jo1]), then $N \subset M$ may or may not have property Γ. We give explicit examples for both cases.

2. Construction of examples

We showed in [Bi1] that if N is a II_1 factor which has property Γ and G is an amenable group acting freely on N, then the inclusion $N \subset N \rtimes G$ has property Γ. Note that if N is the hyperfinite II_1 factor, we can make any countable discrete group G act on N in this way (via the Bernoulli shift action). Thus taking G to be a finite group or an infinite amenable group, for instance $G = S_\infty$, the group of finite permutations of an infinite countable set, we get examples of inclusions of property Γ factors N, M of finite index (G finite) or infinite index (G infinite), resp., such that $N \subset M$ also has property Γ.

Another class of examples comes from *strongly amenable* inclusions of hyperfinite II_1 factors $N \subset M$ with finite Jones's index $[M : N]$ [Po3, Po7], in particular, *finite depth* subfactors [Oc2, Po2] (see also [Ka1, Ha, HSch, Oc3, We1, We2]) of R, the hyperfinite II_1 factor. In this case $N \subset M$ is completely classified by the associated sequence of higher relative commutants $\{N_k' \cap N \subset N_k' \cap M\}_{k \in \mathbb{N}}$ since $N \subset M \cong \overline{\bigcup N_k' \cap N}^w \subset \overline{\bigcup N_k' \cap M}^w$. But the pair $N \bar{\otimes} R \subset M \bar{\otimes} R$ enjoys the same properties (it has the same higher relative commutants classifying it), in other words, $N \subset M$ is stable, i.e., $N \subset M \cong N \bar{\otimes} R \subset M \bar{\otimes} R$. In [Bi1] we showed that this is equivalent to the fact that the algebra $M' \cap N^\omega$ is noncommutative, in particular, $N \subset M$ has property Γ.

The standard technique of constructing subfactors of the hyperfinite II_1 factor is as follows: construct an inclusion of four finite-dimensional C^*-algebras

$$(*) \qquad \begin{array}{ccc} B_0 & \subset & B_1 \\ \cup & & \cup \\ A_0 & \subset & A_1, \end{array}$$

satisfying the *commuting square* condition [PiPo, Po2, HSch, GHJ, We1], i.e., $E_{B_0} E_{A_1} = E_{A_0}$, where E_{B_i} and E_{A_i} denote the trace-preserving conditional expectation onto B_i, resp., A_i. Let $B_0 \subset B_1 \subset B_2 := \langle B_1, e_{B_0} \rangle$ be the Jones basic construction [Jo1], and suppose the initial commuting square $(*)$ is *symmetric* [HSch] or *nondegenerate* [Po7], i.e., $A_0 \subset A_1 \subset \langle A_1, e_{B_0} \rangle$ is (isomorphic to) the basic construction for $A_0 \subset A_1$. Iterating the basic construction gives then a sequence of commuting squares

$$\begin{array}{ccc} B_n & \subset & B_{n+1} \\ \cup & & \cup \\ A_n & \subset & A_{n+1}, \end{array}$$

and one obtains an inclusion of hyperfinite II_1 factors with finite Jones' index in the usual way as $N := \overline{\bigcup_n A_n}^w \subset M := \overline{\bigcup_n B_n}^w$ by performing the GNS construction with respect to the Markov trace on $\bigcup_n B_n$. The inclusions $N \subset M$ constructed in this way have property Γ (Definition 1.1) since the Jones projections $\{e_{B_n}\}$ form a nontrivial central sequence for M contained in N. Taking the noncommuting central sequences $\{e_{B_n}\}$ and $\{e_{B_{n+1}}\}$ one sees that these inclusions actually have the relative Mc Duff property [Bi1]. In particular, all the

examples of irreducible subfactors of the hyperfinite II_1 factor with index > 4 constructed by Haagerup and Schou [HSch] have property Γ in the sense of Definition 1.1 (or stronger, the relative Mc Duff property in the sense of [Bi1]). Note that Haagerup proved in recent work [Ha] that all infinite depth subfactors of R with index between 4 and $3+\sqrt{3}$ have principal graph A_∞ and are therefore nonamenable in the sense of [Po7]. As pointed out by Popa, it would be interesting to study centrally trivial automorphisms and compute the analogue of Connes's invariant $\chi(M)$ as introduced in [Ka2] for these subfactors, since this invariant may lead to a distinction of nonamenable subfactors of the hyperfinite II_1 factor with same higher relative commutant invariant.

Now let us give an example of a subfactor of the hyperfinite II_1 factor R of infinite index that does not contain nontrivial central sequences for the ambient algebra [Po4].

Lemma 2.1. *Let M be a finite von Neumann algebra, $N \subset M$ a subalgebra, and suppose there is a unitary $u \in M$ such that $uNu^* \perp N$. Then N does not contain nontrivial central sequences for M; in other words, $N \subset M$ does not have property Γ.*

Proof (see [Po4]). Let $(x_n)_{n\in\mathbb{N}} \subset N$ be a central sequence for M. Replacing x_n by $x_n - \tau(x_n)$ we may assume that $\tau(x_n) = 0$. Then $\|ux_nu^* - x_n\|_2 \to 0$ as $n \to \infty$. We compute then $\|ux_nu^* - x_n\|_2^2 = 2\|x_n\|_2^2$, which implies that $(x_n)_{n\in\mathbb{N}}$ is trivial. $\square$

Now it is not difficult to construct a subfactor of R satisfying the conditions of Lemma 2.1. All we need to do is to take an amenable I.C.C. (infinite conjugacy classes) group G and an I.C.C. subgroup H such that $gHg^{-1} \cap H = \{e\}$ for some $g \in G$. Then $\lambda(g)L(H)\lambda(g)^{-1} \perp L(H)$ (λ denotes as usual the left regular representation of G). Hence the group von Neumann algebra $L(H) \subset L(G)$ contains only trivial central sequences for $L(G)$; i.e., $L(H) \subset L(G)$ does not have property Γ. We give explicit examples of such groups (see [Po4]). Consider the group H of affine transformations over $\mathbb{Q}$, which is amenable and I.C.C. Let $S_\infty H$ be the group of all permutations of H and SH the subgroup of finite permutations. Identify H with the subgroup of translations in $S_\infty H$, and let G be the group generated by SH and H in $S_\infty H$. Then $G/SH \cong H$. Since SH is amenable, this implies that G is amenable, and it is easy to see that $gHg^{-1} \cap H = \{e\}$ for all $g \in G\backslash H$. This implies that G is I.C.C.

A different class of examples arises from the subfactors $N_G \subset M$ as introduced in [Po1] and studied in [Bi2]. We recall the construction. Let R be the hyperfinite II_1 factor and G be a finitely generated countable discrete group acting on R via σ, and fix a generating set $\{g_1, \dots, g_n\} \subset G$. Take automorphisms $\theta_i \in \operatorname{Aut} R$, $0 \le i \le 2n$, such that $\theta_0 = \mathrm{id}$, $\theta_i = \sigma_{g_i} \bmod \operatorname{Int} R$, $1 \le i \le n$, and $\theta_i = \sigma_{g_i}^{-1} \bmod \operatorname{Int} R$, $n+1 \le i \le 2n$. Set

$$N_G = \left\{ \sum_{i=0}^{2n} \theta_i(x) e_{ii} \mid x \in R \right\} \subset M = M_{2n+1\times 2n+1}(R) = R \bar{\otimes} M_{2n+1\times 2n+1}(\mathbb{C}),$$

where $\{e_{ij}\}_{0\le i,j\le 2n}$ denote the usual matrix units in $M_{2n+1\times 2n+1}(\mathbb{C})$. Note that $N_G \subset M$ is an inclusion of hyperfinite II_1 factors of index $[M : N_G] = (2n+1)^2$ and nontrivial relative commutant $N_G' \cap M \neq \mathbb{C}$. We prove

Theorem 2.2. *Let G, σ, N_G, and M be as above. Then $N_G \subset M$ has property Γ if and only if $R \subset R \rtimes_\sigma G$ has property Γ.*

The proof uses the following three lemmas.

Lemma 2.3. *$N_G \subset M$ has property Γ if and only if given any $\epsilon > 0$ and $x_1, \dots, x_m \in R$, there is a unitary $u \in R$, $\tau(u) = 0$, such that*

$$\|ux_r - x_r\theta_i^{-1}\theta_j(u)\|_2 \le \epsilon,$$

for all $1 \le r \le m$, $0 \le i, j \le 2n$.

Proof. The proof is straightforward and left as an exercise to the reader. □

Denote by $\{u_g\}_{g\in G}$ the unitaries that implement the action σ of G on R, i.e., $u_g x u_g^* = \sigma_g(x)$, $x \in R$. Since the $*$-algebra $\{\sum_{\text{finite}} x_g u_g \mid x_g \in R\}$ is $\|\cdot\|_2$ dense in $R \rtimes_\sigma G$, the following lemma is trivial.

Lemma 2.4. *$R \subset R \rtimes_\sigma G$ has property Γ if and only if given any $\epsilon > 0$, $x_1, \dots, x_m \in R$, and $u_{h_1}, \dots, u_{h_l} \in R \rtimes_\sigma G$, there is a unitary $u \in R$, $\tau(u) = 0$, such that*

$$\|x_i u - u x_i\|_2 \le \epsilon, \quad 1 \le i \le m, \qquad \|\sigma_{h_i}(u) - u\|_2 \le \epsilon, \quad 1 \le i \le l.$$

We need another lemma.

Lemma 2.5. *Let G be generated by $\{g_i,\ 1 \le i \le 2n\}$, where $g_{n+i} = g_i^{-1}$, $1 \le i \le n$, and let $g_0 = e$, the identity of G. Fix some $h \in G$, and let $h = g_{i_1} \cdots g_{i_r}$ be some expression of h in terms of the g_i's. Then $\|\sigma_{g_i}(x) - x\|_2 \le \epsilon$, $0 \le i \le 2n$, implies that $\|\sigma_h(x) - x\|_2 \le r\epsilon$.*

Proof. Use induction on the number of generators that appear in some expression of the element $h \in G$. □

Proof of Theorem 2.2. First note that there are unitaries $u_i \in R$ with $\sigma_{g_i}(x) = u_i\theta_i(x)u_i^*$, $x \in R$, $0 \le i \le 2n$.

Suppose now that $N_G \subset M$ has propery Γ, and let $\epsilon > 0$, $x_1, \dots, x_m \in R$ be given. By Lemma 2.3 there is a unitary $u \in R$, $\tau(u) = 0$, such that

$$\|ux_r - x_r\theta_i^{-1}\theta_j(u)\|_2 \le \epsilon, \qquad 0 \le i, j \le 2n, \ 1 \le r \le m, \tag{1}$$

$$\|uu_s - u_s\theta_i^{-1}\theta_j(u)\|_2 \le \epsilon, \qquad 0 \le i, j \le 2n, \ 0 \le s \le 2n. \tag{2}$$

In particular, choosing $i = j$ in (1) we have $\|ux_r - x_r u\|_2 \le \epsilon$, $1 \le r \le m$. Furthermore, (2) implies with $i = 0$ that $\|uu_s - u_s\theta_j(u)\|_2 \le \epsilon$; i.e., if $s = j$, we get

$$\|u - u_j\theta_j(u)u_j^*\|_2 = \|u - \sigma_{g_j}(u)\|_2 \le \epsilon, \qquad 0 \le j \le 2n.$$

Choosing ϵ small enough from the start we get that $R \subset R \rtimes_\sigma G$ has property Γ from Lemma 2.5 and then Lemma 2.4.

Conversely, assume $R \subset R \rtimes_\sigma G$ has property Γ. Let $x_1, \dots, x_m \in R$, $\epsilon > 0$, and set $c = \sup_{1\le r\le m} \|x_r\|$. By Lemma 2.4 we can find a unitary $u \in R$,

$\tau(u) = 0$, such that

$$\|x_r u - u x_r\|_2 \leq \frac{\epsilon}{3}, \qquad \|\sigma_{g_i^{-1} g_j}(u) - u\|_2 \leq \frac{\epsilon}{3c},$$

$$\|\theta_j^{-1}\theta_i(u_i^*)\theta_j^{-1}(u_j)u - u\theta_j^{-1}\theta_i(u_i^*)\theta_j^{-1}(u_j)\|_2 \leq \frac{\epsilon}{3c}$$

for all i, j, r.

Thus

$$\begin{aligned}
&\|ux_r - x_r\theta_i^{-1}\theta_j(u)\|_2 \\
&\quad \leq \|ux_r - x_r u\|_2 + c\|u - \sigma_{g_i^{-1}g_j}(u)\|_2 + c\|\sigma_{g_i^{-1}g_j}(u) - \theta_i^{-1}\theta_j(u)\|_2 \\
&\quad \leq \frac{2\epsilon}{3} + c\|u_i^*\theta_i^{-1}(u_j)\theta_i^{-1}\theta_j(u)\theta_i^{-1}(u_j)^* u_i - \theta_i^{-1}\theta_j(u)\|_2 \\
&\quad = \frac{2\epsilon}{3} + c\|u_i^*\theta_i^{-1}(u_j)\theta_i^{-1}\theta_j(u) - \theta_i^{-1}\theta_j(u)u_i^*\theta_i^{-1}(u_j)|_2 \\
&\quad = \frac{2\epsilon}{3} + c\|\theta_i(u_i^*)u_j\theta_j(u) - \theta_j(u)\theta_i(u_i^*)u_j\|_2 \\
&\quad = \frac{2\epsilon}{3} + c\|\theta_j^{-1}\theta_i(u_i^*)\theta_j^{-1}(u_j)u - u\theta_j^{-1}\theta_i(u_i^*)\theta_j^{-1}(u_j)\|_2 \\
&\quad = \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \quad \square
\end{aligned}$$

We can now use Theorem 2.2 to construct examples of subfactors of the hyperfinite II_1 factor of finite index which do not contain nontrivial central sequences for the ambient algebra. Let G be a group with property T (which is then necessarily finitely generated), for instance $\mathrm{Sl}(n, \mathbb{Z})$, $n \geq 3$, that acts on R via the Bernoulli shift action σ, a properly outer and ergodic action. Then we have

Lemma 2.6. *$R \subset R \rtimes_\sigma G$ does not have property Γ.*

Proof. It was shown in [Ch1] that the crossed product $R \rtimes_\sigma G$ does not have property Γ; hence, the inclusion $R \subset R \rtimes_\sigma G$ cannot have property Γ. $\square$

Theorem 2.2 implies then that the inclusion of hyperfinite II_1 factors $N_G \subset M$ does not have property Γ. This construction gives a large class of examples of inclusions of hyperfinite II_1 factors not having property Γ, since there are uncountably many mutually nonconjugate properly outer cocycle actions of an infinite property T group on R (see [Po1, Po5, Po6]). Note that $N_G \subset M$ is amenable (in the sense of [Po7]) if and only if the group G is amenable. We obtain therefore in this way nonamenable subfactors of the hyperfinite II_1 factor which do not have property Γ as in Definition 1.1.

Remark 2.7. Popa defines in [Po5] the notion (different from [Po7]!) of *amenability* of an inclusion of II_1 factors $N \subset M$. In his terminology, if $N \subset M = N \rtimes G$ then $N \subset M$ is amenable if and only if G is amenable. In this case amenability of $N \subset M$ implies that $N \subset M$ has property Γ. However, in general these two notions are different. Popa shows that if $[M : N] < \infty$, then $N \subset M$ is amenable. We have seen above that in finite index we may have both property Γ and nonproperty Γ.

There are still a number of open problems related to the notion of property Γ of an inclusion. For instance, since property Γ of a single II_1 factor has a nice characterizaton in terms of M-M-bimodules (see [Po5]), it would be interesting to find such a characterization also for the property Γ of an inclusion $N \subset M$.

Finally, let us remark that central sequences in crossed products have been studied extensively by Bédos, Choda, Jones, Jones and Hermann, Ocneanu, and Philipps [Ch1, Ch2, Be, Jo2, JoH, Oc1, Ph].

References

[Bi1] D. Bisch, *On the existence of central sequences in subfactors*, Trans. Amer. Math. Soc. **321** (1990), 117–128.

[Bi2] ——, *Entropy of groups and subfactors*, J. Funct. Anal. **103** (1992), 190–208.

[Be] E. Bédos, *On actions of amenable groups on* II_1 *factors*, J. Funct. Anal. **91** (1990), 404–414.

[Ch1] M. Choda, *Property T and fullness of the group measure space construction*, Math. Japon. **27** (1982), 535–539.

[Ch2] ——, *Crossed products and property T*, Math. Japon. **26** (1981), 557–567.

[GHJ] F. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter graphs and towers of algebras*, Math. Sci. Res. Inst. Publ., vol. 14, Springer, New York, 1989.

[Ha] U. Haagerup, in preparation.

[HSch] U. Haagerup and J. Schou, *Some new subfactors of the hyperfinite* II_1 *factor*, preprint, 1989.

[Jo1] V. F. R. Jones, *Index of subfactors*, Invent. Math. **72** (1983), 1–25.

[Jo2] ——, *Central sequences in crossed products of full factors*, Duke Math. J. **49** (1982), 29–33.

[JoH] R. Herman and V. F. R. Jones, *Central sequences in crossed products*, Contemp. Math., vol. 62, Amer. Math. Soc., Providence, RI, 1987, pp. 539–544.

[Ka1] Y. Kawahigashi, *On flatness of Ocneanu's connections on the Dynkin diagrams and classification of subfactors*, preprint, 1990.

[Ka2] ——, *Centrally trivial automorphisms and an analogue of Connes'* $\chi(M)$ *for subfactors*, preprint, 1992.

[Oc1] A. Ocneanu, *Actions of discrete amenable groups on von Neumann algebras*, Lecture Notes in Math., vol. 1138, Springer, New York, 1985.

[Oc2] ——, *Quantized groups string algebras and Galois theory for operator algebras*, Operator Algebras and Applications 2, London Math. Soc. Lecture Notes Ser., vol. 136, Cambridge Univ. Press, Cambridge, 1988, pp. 119–172.

[Oc3] ——, *Quantum symmetry, differential geometry of finite graphs and classification of subfactors*, University of Tokyo Seminary Notes 45, 1991 (notes recorded by Y. Kawahigashi).

[Ph] J. Philipps, *Automorphisms of full* II_1 *factors with applications to factors of type* III, Duke Math. J. **43** (1976), 375–385.

[PiPo] M. Pimsner and S. Popa, *Entropy and index of subfactors*, Ann. Sci. École Norm. Sup. (4) **19** (1986), 57–106.

[Po1] S. Popa, *Sousfacteurs, actions des groupes et cohomologie*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), 771–776.

[Po2] ——, *Classification of subfactors: reduction to commuting squares*, Invent. Math. **101** (1990), 19–43.

[Po3] ——, *Sur la classification des sous-facteurs d'indice fini du facteur hyperfini*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 95–100.

[Po4] ——, *Orthogonal pairs of* $*$*-subalgebras in finite von Neumann algebras*, J. Operator Theory **9** (1983), 253–268.

[Po5] ——, Correspondences, INCREST preprint, 1986.

[Po6] ——, *Some rigidity results in type* II_1 *factors*, C. R. Acad. Sci. Paris Sér. I Math. (to appear).

[Po7] ——, *Classification of amenable subfactors and their automorphisms*, preprint, 1991.

[MvN] F. Murray and J. von Neumann, *Rings of operators*. IV, Ann. of Math. **44** (1943), 716–808.

[We1] H. Wenzl, *Hecke algebras of type A and subfactors*, Invent. Math. **92** (1988), 345–383.

[We2] ——, *Quantum groups and subfactors of type B, C and D*, Comm. Math. Phys. **133** (1990), 383–432.

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