BORSUK'S ANTIPODAL AND FIXED-POINT THEOREMS
FOR SET-VALUED MAPS

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Abstract. The purpose of this paper is to obtain the extensions of Borsuk's antipodal and fixed-point theorems for set-valued maps.

Borsuk [3] related the following relative results:

**Borsuk's antipodal theorem.** A single-valued antipodal-preserving continuous map \( f: S^n \to E^n \) has a zero value.

**Borsuk's fixed-point theorem.** Let \( U \) be a bounded symmetric convex open neighborhood of the origin in \( E^n \), and let \( f: \overline{U} \to E^n \) be antipodal-preserving on \( \partial U \), i.e., \( -f(a) = f(-a) \) for each \( a \in \partial U \). Then \( f \) has a fixed point.

In this note we consider the set-valued maps. With a proper definition of antipodality, we prove that Borsuk's antipodal theorem can be extended for set-valued maps on the boundary of a symmetric bounded balanced neighborhood of the origin. Such boundaries are more general than the homeomorphic image of spheres. At the same time we also prove Borsuk's fixed-point theorem on a symmetric balanced set of a locally convex space. The main results are summarized in Theorems 4, 5, 6, and 7. Lemmas 1 and 2 are two approximation properties which play a crucial role in this paper.

Let us first recall some definitions. Let \( X \) and \( Y \) be two topological spaces; capital letters \( F: X \to Y \) denote nonempty set-valued maps while noncapital letters \( f: X \to Y \) will denote single-valued functions. \( F \) is called upper semi-continuous (u.s.c.) if for each open set \( G \) of \( Y \) the set \( \{ x \in X | F(x) \subset G \} \) is open in \( X \). \( F \) is said to have open lower sections if, for each \( y \in Y \), \( F^{-1}(y) = \{ x \in X | y \in F(x) \} \) is open in \( X \). The set \( \text{Gr} \, F \) is the graph of \( F \) which is the set \( \{(x, y)|y \in F(x), x \in X\} \). When \( Y \) is a topological vector space and \( S \subset Y \), \( \text{co} \, S \) denotes the convex hull of \( S \). A set \( B \subset Y \) is said to be balanced if \( rB \subset B \) for every real number \( r \) with \( |r| \leq 1 \).

Let \( E \) be the normed space of all those sequences \( x = (x_1, x_2, \ldots) \) of real numbers having at most finitely many \( x_n \neq 0 \), with the norm \( ||x|| = \sum |x_i| \). The subset \( \{ x \in E | x_i = 0 \text{ for all } i > n \} \) is denoted by \( E^n \); the unit \( n \)-sphere \( S^n = \{ x \in E^{n+1} | ||x|| = 1 \} \). Let \( X \) and \( Y \) be subsets of \( E \), \( S \subset X \) be a
subset which is symmetric with respect to the origin, and \( F: X \to Y \). We say that \( F \) is antipodal on \( S \) if \( F(x) \cap (-F(-x)) \neq \emptyset \) for all \( x \in S \). This is a generalization of the original single-valued functions to set-valued maps.

The following lemma is a generalization of a theorem of Cellina (see [1, Lemma 13.1, p. 67]). For different versions of the statement, we refer to [2, Theorem 1, p. 19].

**Lemma 1.** Let \( X \) be a compact set and \( \mathcal{B}_1 \) an open symmetric neighborhood base at 0 in a topological vector space \( E_1 \). Let \( Y \) be a set and \( \mathcal{B}_2 \) an open convex neighborhood base at 0 in a locally convex space \( E_2 \). Suppose the correspondence \( F: X \to Y \) is u.s.c. with convex values. For \( V \in \mathcal{B}_1 \) define \( F^V: X + V \to E_2 \) via \( F^V(x) = \text{co}\bigcup_{z \in (x+V) \cap X} F(z) \). Then for each \( W_1 \in \mathcal{B}_1 \) and \( W_2 \in \mathcal{B}_2 \) there exists \( V \in \mathcal{B}_1 \) such that

\[
\text{Gr} F^V \subset \text{Gr} F + W_1 \times W_2
\]

and \( F^V \) has open lower sections.

Furthermore, if \( S \subset X \) is symmetric with the origin and \( F \) is antipodal on \( S \), then \( F^V \) is antipodal on \( S + V \).

**Proof.** We choose \( V_1 \in \mathcal{B}_1 \) such that \( V_1 + V_1 + V_1 \subset W_1 \). For any \( x \in X \) there exists \( V_x \in \mathcal{B}_1 \) such that \( \bigcup_{z \in x+V_x} F(z) \subset F(x) + W_2 \). We choose \( U_x \in \mathcal{B}_1 \) such that \( U_x + U_x + U_x \subset V_x \).

Since \( X \) is compact, there exists a finite cover \( x_1 + (U_{x_1} \cap V_1), \ldots, x_n + (U_{x_n} \cap V_1) \) of \( X \). Let \( V = (\bigcap_{i=1}^n U_{x_i}) \cap V_1 \). Fix \( x \in X + V \). Then \( x \in x_i + (U_{x_i} \cap V_1) + V \) for some \( i \). Hence \( x + V \subset x_i + (U_{x_i} \cap V_1) + V + V \subset x_i + (V_{x_i} \cap W_1) \). Thus \( \bigcup_{z \in (x+V) \cap X} F(z) \subset \bigcup_{z \in (x_i+V_{x_i})} F(z) \subset F(x_i) + W_2 \). Since \( F \) has convex values and \( W_2 \) is convex, \( \text{co}\bigcup_{z \in (x+V) \cap X} F(z) \subset F(x_i) + W_2 \), and then \( \{x\} \times F^V(x) \subset \{(x_i) + W_1\} \times (F(x_i) + W_2) \) for such \( i \). Thus \( \text{Gr} F^V \subset \text{Gr} F + W_1 \times W_2 \).

Let \( \text{Gr} H = \text{Gr} F + V \times \{0\} \). Then \( H: X + V \to E_2 \) has open lower sections obviously, and hence \( \text{co}\text{Gr} H: X + V \to E_2 \) has open lower sections where \( \text{co}H(x) = \text{co}(H(x)) \) for each \( x \in X + V \). For each \( x \in X + V \), \( \text{co}(H(x)) = \text{co}\bigcup_{z \in (x+V) \cap X} F(z) = F^V(x) \). This implies that \( F^V \) has open lower sections.

For any \( x \in S + V \) there exist \( x' \in S, v \in V \) such that \( x = x' + v \). Since \( S \) and \( V \) are symmetric with the origin, \( -x = -x' + (-v) \in (S + V) \). Since \( F \) is antipodal on \( S \), \( \emptyset \neq F(x') \cap (-F(-x')) \subset F^V(x) \cap (-F^V(\neg x)) \). Hence \( F^V \) is antipodal on \( S + V \). This completes the proof.

**Lemma 2.** Let \( U \) be an open symmetric balanced neighborhood of the origin in \( E^{n+1} \). Define \( g: U \setminus \{0\} \to E^{n+1} \) by \( g(x) = d(x, \partial U)x/\|x\| + x \). Then \( g \) is continuous and symmetric; i.e., \( g(-x) = -g(x) \). Furthermore, for any \( \varepsilon > 0 \) there exists \( k \) such that

\[
d(g^k(U \setminus \{0\}), \partial U) = \sup\{d(g^k(x), \partial U) | x \in U \setminus \{0\}\} < \varepsilon.
\]

**Proof.** By the property of \( U \), \( d(x, \partial U) \) is continuous and \( d(x, \partial U) = d(-x, \partial U) \) in \( U \setminus \{0\} \). Thus \( g \) is continuous, and \( g(-x) = -g(x) \) for all \( x \in U \setminus \{0\} \); i.e., \( g \) is symmetric.

Let \( r_1 = \inf\{\|x\| | x \in \partial U\} \) and \( r_2 = \sup\{\|x\| | x \in \partial U\} \). For every \( x \in U \setminus \{0\} \) let \( x_I = r_1 x/\|x\|, x_T = r_2 x \), where \( r_x = \inf\{r \|rx/\|x\| \in \partial U\} \). Then
\[ r_1 \leq \|g(x)\| = d(x, \partial U) + \|x\| \leq r_x, \text{ and hence } g(x) = \|g(x)\|x/\|x\| \in \overline{xtx_T}, \text{ where } \overline{xtx_T} \text{ denotes the line segment from } x_t \text{ to } x_T. \text{ Since } g^j(x) = d(g^{j-1}(x), \partial U)x/\|x\| + \|g^{j-1}(x)\|x/\|x\|, \text{ it follows that } \|g^{j-1}(x)\| \leq \|g^j(x)\| = d(g^{j-1}(x), \partial U) + \|g^{j-1}(x)\| \leq r_{g^{j-1}(x)} = r_x \text{ and } g^j(x) \in g^{j-1}(x)\overline{xtx_T} \text{ for each } j = 2, 3, \ldots. \]

Fix \( x \in U\setminus\{0\} \), where \( \|x\| \geq r_1 \). If \( d(x, \partial U) = \delta \), then there is \( p_x \in \partial U \) such that \( \|x - p_x\| = \delta \). If \( p_x = x_T \), then for each \( x' \in \overline{xx_T}, \ d(x', \partial U) < \delta \). If \( p_x \neq x_T \), then each \( x' \in \overline{xx_T} \), the line \( l \) passing through \( x' \) and parallel to \( \overline{xpx_T} \), intersects \( \overline{xpx_T} \). Then \( p_x' \notin U \) and \( d(x', \partial U) < \|x' - p_x\| = \|x'\|\delta/\|x\| \leq r_2\delta/r_1 \). Hence, if \( \|x\| \geq r_1 \) and \( d(x, \partial U) < r_1\epsilon/r_2 \), then \( d(\overline{xx_T}, \partial U) < \epsilon \). Then \( d(g^j(x), \partial U) < \epsilon \) for \( j = 0, 1, 2, \ldots \).

Fix \( y \in U\setminus\{0\} \). Then \( \|g(y)\| \geq r_1 \), and \( g^k(y) = d(g^{k-1}(y), \partial U)y/\|y\| + g^{k-1}(y) \)
\[ = [d(g^{k-1}(y), \partial U) + d(g^{k-2}(y), \partial U) + \ldots + d(g(y), \partial U)]y/\|y\| + g(y) \]
\[ = [d(g^{k-1}(y), \partial U) + \ldots + d(g(y), \partial U) + \|g(y)\|]y/\|y\|. \]
Since
\[ \|g^k(y)\| = d(g^{k-1}(y), \partial U) + \ldots + d(g(y), \partial U) + \|g(y)\| \leq r_2 \]
and \( \|g(y)\| \geq r_1 \), \( d(g^{k-1}(y), \partial U) + \ldots + d(g(y), \partial U) \leq r_2 - r_1 \), and hence there is some \( i \) where \( 1 \leq i \leq k - 1 \) such that \( d(g^i(y), \partial U) \leq (r_2 - r_1)/(k - 1) \). If \( k > (r_2 - r_1)r_1 \epsilon/2, \) then \( d(g^i(y), \partial U) < (r_2 - r_1)\epsilon/[r_2 - r_1] = r_1\epsilon/r_2 < \epsilon \), and hence
\[ d(g^k(y), \partial U) = d(g^{k-i}(g^i(y)), \partial U) < \epsilon. \]
This completes the proof.

**Lemma 3.** Let \( E_1 \) and \( E_2 \) be two topological vector spaces. Let \( S \subset E_1 \) and \( S \) be compact and symmetric with respect to \( 0 \). Let \( F: S \rightarrow E_2 \) be antipodal-preserving, convex-valued, and with open lower sections. Then \( F \) has a single-valued continuous antipodal selection \( f: S \rightarrow E_2 \).

**Proof.** By definition, \( \{F^{-1}(y) \cap [-F^{-1}(-y)] \mid y \in E_2 \} \) is an open cover of \( S \); hence, there is a finite subcover \( \Gamma = \{F^{-1}(y_i) \cap [-F^{-1}(-y_i)] \mid i = 1, 2, \ldots, n \} \). Let \( \{\phi_i\}_{i=1}^n \) be a partition of unity of \( S \) with respect to \( \Gamma \). It implies that each \( \phi_i \) is nonnegative continuous on \( S \), \( \sum_{i=1}^n \phi_i(x) = 1 \), and \( \phi_i(x) > 0 \Rightarrow x \in F^{-1}(y_i) \cap [-F^{-1}(-y_i)] \). Define \( p: S \rightarrow E_2 \) by
\[ p(x) = \frac{\sum_{i=1}^n \phi_i(x)y_i - \sum_{i=1}^n \phi_i(-x)y_i}{\sum_{i=1}^n \phi_i(x) + \sum_{i=1}^n \phi_i(-x)}. \]
Then \( p \) is continuous, and \( -p(-x) = p(x) \) for all \( x \in S \). If \( \phi_i(x) \neq 0 \), then \( x \in F^{-1}(y_i) \), and hence \( y_i \in F(x) \). If \( \phi_i(-x) \neq 0 \), then \( -x \in -F^{-1}(-y_i) \), and hence \( -y_i \in F(x) \). Since \( F \) is convex-valued, \( p(x) \in F(x) \) for all \( x \in S \).

Borsuk antipodal and Borsuk-Ulam theorems [3, §4, Theorem 5.2, p. 44] can be generalized to the following two theorems.

**Theorem 4.** Let \( U \) be an open bounded symmetric neighborhood of the origin in \( E_1 \), and let \( F: \partial U \rightarrow E_2 \) be u.s.c., closed, convex-valued, and antipodal-preserving. Then \( F \) has a zero.
Proof. Let \( \mathbf{0} \) denote the zero map from \( \partial U \) into \( \{0\} \). Suppose that \( F \) does not have zero. Then by [4, Theorem 1] there are open convex neighborhoods \( W \) of 0 in \( E^n \) and \( W_1 \) of 0 in \( E^{n+1} \) such that \( \text{Gr} F + W_1 \times W \) is empty. By Lemma 1 there is an open neighborhood \( V \) of 0 in \( E^{n+1} \) such that \( \text{Gr} F^V \subset \text{Gr} F + W_1 \times W \) and the domain of \( F^V \) is \( \partial U + V \). Let \( V_1 \) be a closed bounded symmetric balanced neighborhood of 0 in \( E^{n+1} \) such that \( V_1 + V_1 \subset V \) and \( F^V|_{\partial U + V_1} \) denote the restriction of \( F^V \) on \( \partial U + V_1 \). Choose \( \lambda, 0 < \lambda < 1 \), such that \( \lambda S_n \subset U \). Define \( g \) on \( U \setminus \{0\} \) as Lemma 2. Then there exists a positive integer \( k \) such that \( g^k(x) \in \partial U + V_1 \) for all \( x \in \lambda S_n \). Let \( K = g^k(\lambda S_n) \). By Lemma 1, \( F^V|_{\partial U + V_1} \) is antipodal and has open lower sections. By Lemma 3, \( F^V|_K \) has a single-valued continuous antipodal selection \( h \). Define \( f: S_n \to E^n \) by \( f(x) = h(g^k(\lambda x)) \). Then \( f \) is a single-valued continuous antipodal map with zero free. This is a contradiction to Borsuk's antipodal theorem. This completes the proof.

**Theorem 5.** Let \( U \) be an open bounded symmetric neighborhood of the origin in \( E^{n+1} \). Then every u.s.c. closed convex-valued map \( F: \partial U \to E^n \) has an intersection point for at least one pair of antipodal points.

Proof. Define \( G: \partial U \to E^n \) by \( G(x) = F(x) - F(-x) \). Then \( G \) is a u.s.c. closed convex-valued antipodal-preserving map. By Theorem 4, \( 0 \in F(x) - F(-x) \), and hence \( F(x) \cap F(-x) \neq \emptyset \) for some \( x \in \partial U \). This completes the proof.

**Theorem 6.** Let \( U \) be an open bounded symmetric balanced neighborhood of the origin in \( E^{n+1} \). Let \( F: \overline{U} \to E^{n+1} \) be u.s.c., closed, convex-valued, and antipodal on \( \partial U \); i.e., \( F(a) \cap [-F(-a)] \neq \emptyset \) for each \( a \in \partial U \). Then \( F \) has a zero value and a fixed point on \( U \).

Proof. Suppose that \( F \) has no zero value. Let \( \mathbf{0} \) denote the zero map from \( \overline{U} \) into \( \{0\} \). Then \( \text{Gr} F \cap \text{Gr} \mathbf{0} = \emptyset \). By [4, Theorem 1] there is a closed convex neighborhood \( W \) of 0 in \( E^{n+1} \) such that \( \text{Gr} F + W \times W \) is empty. By Lemma 1 there exists a symmetric neighborhood \( V \) of 0 such that \( \text{Gr}(F^V) \subset \text{Gr} F + W \times W \). Hence \( 0 \notin F^V(x) \) for all \( x \in \overline{U} + V \). By Lemma 3, \( F^V|_{\overline{U}} \) has a single-valued selection \( f \) such that \( f \) is antipodal on \( \partial U \).

Define \( J: \overline{U} \to E^{n+2} \) by \( J(x) = x + d(x, \partial U)\mathbb{E}_{n+2} \). Then \( J \) is one-to-one continuous, and \( J(x) = x \) on \( \partial U \). Let \( V = \{x + ud_{n+2} \mid x \in U, |u| < d(x, \partial U)\} \). Then \( V \) is open, bounded, symmetric, and balanced in \( E^{n+2} \) and \( \partial V = J(\overline{U}) \cup \{-J(\overline{U})\} \).

Define \( H: \partial V \to E^{n+1} \) by
\[
H(x) = \begin{cases} 
  f(J^{-1}(x)) & \text{if } x \in J(\overline{U}), \\
  -f(J^{-1}(-x)) & \text{if } x \in -J(\overline{U}).
\end{cases}
\]

Then \( H \) is a continuous antipodal single-valued function with zero free. This is a contradiction to Theorem 4. Thus \( F \) has a zero value.

By the above conclusion, \( F - I \) has a zero value; i.e., there exists \( x \in \overline{U} \) such that \( 0 \in F(x) - \{x\} \). Hence \( x \in F(x) \) for some \( x \in \overline{U} \). This proves the theorem.

The following theorem is a generalization of a result of Borsuk [3, §4, Theorem 3.3, p. 57].
Theorem 7. Let $M$ be a closed bounded symmetric balance set at 0 in a locally convex space $E$. Let $F: M \rightarrow E$ be a u.s.c. closed convex-valued map such that the closure of $F(M)$ is compact and $F$ is antipodal on the boundary of $M$. Then $F$ has at least one fixed point.

Proof. Let $\mathcal{B}$ denote a closed bounded symmetric convex neighborhood base at 0 in $E$. For each $V \in \mathcal{B}$ there is a finite subset $S$ of $F(M)$ such that $(y + V) \cap S \neq \emptyset$ for each $y \in F(M)$. Let $S_V$ be a finite subset of $E$ such that $S \subset S_V$. Let $H_{S_V}$ denote the finite-dimensional space spanned by $S_V$.

Define $G_V : M \cap H_{S_V} \rightarrow H_{S_V}$ by $G_V(x) = (F(x) + V) \cap H_{S_V}$. Then $G_V$ is u.s.c. with nonempty compact convex values. Since $F(a) \cap -F(-a) \neq \emptyset$ for all $a \in \partial(M \cap H_{S_V})$, $\emptyset \neq \{(F(a) \cap (-F(-a)) + V) \cap H_{S_V}$ for all $a \in \partial(M \cap H_{S_V})$; i.e., $G_V$ is antipodal on its boundary. If 0 is a relative interior point of $M \cap H_{S_V}$, then by Theorem 6 there is $x_V$ such that $x_V \in G_V(x_V)$; i.e., $x_V \in F(x_V) + V$. On the other hand, if 0 $\in \partial(M \cap H_{S_V})$, then $M \cap H_{S_V} = \partial(M \cap H_{S_V})$. Since $G_V$ is antipodal on $M \cap H_{S_V}$, we have $G_V(0) = -G_V(0)$. Since $G_V(0)$ is convex, we have 0 $\in G_V(0)$.

From the above argument, we get that in any situation for each $V \in \mathcal{B}$ there exist $x_V \in M$ such that $x_V = y_V + v$, where $y_V \in F(x_V)$ and $v \in V$. Since the closure of $F(M)$ is compact, there is a subnet of $\{y_V\}$ that converges to $x_0$. Then $x_0 \in F(x_0)$. This completes the proof.

Now we give the following application of Theorem 4 which is a result of Krein, Krasnoselsky, and Milman [3, §5, Theorem 5.4, p. 80].

Theorem 8. Let $M, N$ be linear subspaces of a Banach space $(E, \|\|)$. If $\dim M > \dim N$, then there is a $x_0 \in M$ such that $d(x_0, N) = \|x_0\| > 0$.

Proof. For each $x \in E$ let $F(x) = \{y \in N \mid d(y, x) = d(x, N)\}$. Then $F(x)$ is compact convex and $x \rightarrow F(x)$ is of closed graph. Furthermore, the map $F : E \rightarrow N$ has the property $F(-x) = -F(x)$ for all $x \in E$. Let $S_M$ denote the unit sphere in $M$. Consequently, applying Theorem 4 to $F|_{S_M} : S_M \rightarrow N$, we obtain a point $x_0 \in S_M$ such that $0 \in F(x_0)$. Clearly, $x_0$ is the required point. This proves the theorem.

References