

## ON A CLASS OF LIPSCHITZ CONTINUOUS FUNCTIONS OF SEVERAL VARIABLES

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(Communicated by Andrew Bruckner)

**ABSTRACT.** We establish an estimate via initial values for functions in a class of Lipschitz continuous functions of several variables. This estimate can be used to investigate the uniqueness of quasi-classical solutions of Cauchy problems for first-order nonlinear partial differential equations (PDEs). Particularly, we give an answer to an open problem posed by S. N. Kružkov.

Let  $T$  be a positive number,  $\Omega_T = (0, T) \times \mathbb{R}^n = \{(t, x) \mid 0 < t < T\}$ ,  $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_n)$ ,  $n \geq 1$ , and  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  be the norm and the scalar product in  $\mathbb{R}^n$ , respectively.

Denote by  $\text{Lip}(\Omega_T)$  the set of all locally Lipschitz continuous functions  $u$  defined on  $\Omega_T$ . Further, set  $\text{Lip}([0, T] \times \mathbb{R}^n) = \text{Lip}(\Omega_T) \cap C([0, T] \times \mathbb{R}^n)$ . For every function  $u$  defined on  $\Omega_T$ , we put

$$\text{Dif}(u) = \{(t, x) \in \Omega_T \mid u \text{ is differentiable at } (t, x)\}.$$

In this note we shall be concerned with the following class of Lipschitz continuous functions:

$$V(\Omega_T) = \{u \in \text{Lip}([0, T] \times \mathbb{R}^n) \mid \exists G \subset [0, T], \text{mes}(G) = 0, \\ \text{Dif}(u) \supset \Omega_T \setminus (G \times \mathbb{R}^n)\}.$$

In other words, a function  $u \in \text{Lip}([0, T] \times \mathbb{R}^n)$  belongs to  $V(\Omega_T)$  iff, for almost all  $t$ ,  $u$  is differentiable at any point  $(t, x)$ .

It is obvious that

$$\text{Lip}([0, T] \times \mathbb{R}^n) \supset V(\Omega_T) \supset C^1(\Omega_T) \cap C([0, T] \times \mathbb{R}^n).$$

Our aim is to prove the following result.

**Theorem 1.** *Let  $u$  be a function in  $V(\Omega_T)$ . If there exist a nonnegative function  $h$  locally bounded on  $\mathbb{R}^n$  and a nonnegative function  $k \in L^1(0, T)$  such that*

$$(1) \quad \left| \frac{\partial u(t, x)}{\partial t} \right| \leq k(t) \cdot [(1 + \|x\|) \|\nabla_x u(t, x)\| + h(x)|u(t, x)|],$$

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Received by the editors October 19, 1992.

1991 *Mathematics Subject Classification.* Primary 26D10.

This work was supported in part by NCSR Vietnam Program "Applied Mathematics" and the National Basic Research Program in Natural Science.

for almost every  $t \in (0, T)$  and for all  $x \in \mathbb{R}^n$ , then

$$(2) \quad |u(t, x)| \leq \exp \left[ C(x) \int_0^t k(\tau) d\tau \right] \cdot \sup_{\|y\| \leq (\|x\|+1) \exp \int_0^t k(\tau) d\tau - 1} |u(0, y)|,$$

where

$$(3) \quad C(x) = \sup \left\{ |h(y)| \mid \|y\| \leq (\|x\| + 1) \exp \int_0^T k(\tau) d\tau - 1 \right\}.$$

**Corollary 1.** Let  $u \in V(\Omega_T)$  and  $u(0, x) \equiv 0$ ,  $x \in \mathbb{R}^n$ . If condition (1) is satisfied for almost every  $t \in (0, T)$  and for all  $x \in \mathbb{R}^n$ , then  $u(t, x) \equiv 0$  in  $\Omega_T$ .

*Remark 1.* Corollary 1, in particular, gives the answer to problem a) posed by S. N. Kružkov in [4]. Theorem 1 and Corollary 1 can be used to investigate the uniqueness of quasi-classical solutions of the Cauchy problems for first-order nonlinear PDEs and the continuous dependence of solutions on initial conditions. In [5, 6] we obtained some results similar to Theorem 1 for subclasses of  $V(\Omega_T)$  and used them to prove the uniqueness of global solutions of the Cauchy problems for nonlinear PDEs of first order.

*Remark 2.* We show by the following example that the Lipschitz continuity of  $u$  is essential in Theorem 1.

Let  $J \subset [0, 1]$  be the Cantor set, i.e., the set of all numbers of the form

$$(C) \quad t = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{3^i},$$

where  $\varepsilon_i$  is either 0 or 2. The set  $J$  is complete, nowhere dense on  $\mathbb{R}^1$ , and  $\text{mes}(J) = 0$ .

We define the function  $v(\cdot)$ , which is called the Cantor ladder, in the following way (see [3]). For  $t \in J$  given by (C), we put

$$v(t) = \sum_{i=1}^{\infty} \frac{b_i}{2^i}, \quad b_i = \frac{\varepsilon_i}{2}.$$

If  $(\alpha, \beta)$  is an open maximum interval in  $(0, 1) \setminus J$ , then  $\alpha, \beta \in J$ ,  $v(\beta) = v(\alpha)$ . We set for  $t \in (\alpha, \beta)$ :  $v(t) = \text{const} = v(\alpha) = v(\beta)$ . It follows that  $v(\cdot) \in C[0, 1]$  and that  $dv/dt(t) = 0$  almost everywhere in  $(0, 1)$ . In fact,  $dv/dt(t) = 0$  for  $t \in (0, 1) \setminus J$ .

Putting  $u(t, x) = v(t)$ ,  $(t, x) \in \Omega_1$ , we easily see that

$$u \in C^1(((0, 1) \setminus J) \times \mathbb{R}^n) \cap C([0, 1] \times \mathbb{R}^n), \quad u(0, x) \equiv 0, \\ \partial u(t, x)/\partial t = 0, \quad \forall (t, x) \in ((0, 1) \setminus J) \times \mathbb{R}^n.$$

The function  $u$  satisfies all the conditions of Theorem 1 and Corollary 1 except Lipschitz continuity. This explains why  $u(t, x) \not\equiv 0$ .

*Proof of Theorem 1.* For an arbitrary point  $(t_0, x_0) \in \Omega_T$ , we have to prove that

$$(2.a) \quad |u(t_0, x_0)| \leq \exp \left[ C(x_0) \int_0^{t_0} k(t) dt \right] \cdot \sup_{\|y\| \leq (\|x_0\|+1) \exp \int_0^{t_0} k(t) dt - 1} |u(0, y)|.$$

Let  $\bar{B}_r = \bar{B}_r^n = \{y \in \mathbb{R}^n \mid \|y\| \leq r\}$ ,  $r \geq 0$ . Denote by  $\Sigma_I(t_0, x_0)$  the set of all absolutely continuous functions  $x(\cdot): I = [0, t_0] \rightarrow \mathbb{R}^n$ , which satisfy almost everywhere in  $I$  the differential inclusion  $dx/dt(t) \in \bar{B}_{k(t) \cdot (\|x(t)\| + 1)}$  subject to the constraint  $x(t_0) = x_0$ .

From Theorem VI-13 in [2], it follows that  $\Sigma_I(t_0, x_0)$  is a nonempty compact set in  $C(I, \mathbb{R}^n)$ . The sets  $Z(t, t_0, x_0) \equiv \{x(t) \mid x(\cdot) \in \Sigma_I(t_0, x_0)\}$  and  $\Gamma(t_0, x_0) \equiv \{(\tau, y) \mid \tau \in I, y \in Z(\tau, t_0, x_0)\}$  are therefore compact sets in  $\mathbb{R}^n$  and  $\mathbb{R}^{n+1}$ , respectively, for all  $t \in I$ . Moreover, by the converse of Ascoli's theorem, the multivalued function  $Z(\cdot, t_0, x_0): I \rightarrow \mathbb{R}^n$  is continuous.

We now define a function  $\varphi(\cdot): I \rightarrow \mathbb{R}^1$  as

$$\varphi(t) = \max\{|u(t, y)| \mid y \in Z(t, t_0, x_0)\}.$$

Then, according to the maximum theorem (see [1, Theorem 1.4.16]), the fact that  $u \in C(\Gamma(t_0, x_0))$  implies  $\varphi(\cdot) \in C(I)$ . In addition, we have:

**Lemma 1.** *For an arbitrary number  $\theta \in (0, t_0)$ ,  $\varphi(\cdot)$  is absolutely continuous on  $[\theta, t_0]$ .*

We shall also need the following:

**Lemma 2.** *We have for every  $t \in I$  the inclusion*

$$(4) \quad Z(t, t_0, x_0) \subset \bar{B}_{(\|x_0\| + 1) \exp \int_t^{t_0} k(\tau) d\tau - 1}.$$

*Proof of Lemma 2.* For every  $\eta > 0$ , put

$$m_\eta(t) = (\|x_0\| + 1 + \eta) \exp \int_t^{t_0} k(\tau) d\tau - 1.$$

The function  $m_\eta(\cdot)$  is absolutely continuous, positive on  $I$  with the derivative  $dm_\eta(t)/dt = -k(t) \cdot (m_\eta(t) + 1)$ . To prove (4) we have only to show that

$$(5) \quad \|x(t)\| < m_\eta(t), \quad \forall t \in I,$$

for all  $x(\cdot) \in \Sigma_I(t_0, x_0)$ ,  $\forall \eta > 0$ .

Since  $m_\eta(t_0) > \|x_0\| = \|x(t_0)\|$ , there exists a number  $\zeta > 0$  such that  $m_\eta(t) > \|x(t)\|$ ,  $\forall t \in (t_0 - \zeta, t_0]$ .

Assume that (5) is false, so that there exists  $t' \in [0, t_0)$  such that  $m_\eta(t') \leq \|x(t')\|$ . Putting  $t_1 = \sup\{t \in [0, t_0) \mid m_\eta(t) \leq \|x(t)\|\} < t_0$ , we have

$$\|x(t_1)\| = m_\eta(t_1), \quad m_\eta(t) > \|x(t)\|, \quad \forall t \in (t_1, t_0]$$

and

$$\begin{aligned} dm_\eta(t)/dt &= -k(t) \cdot (m_\eta(t) + 1) \leq -k(t) \cdot (\|x(t)\| + 1) \\ &\leq -\|dx(t)/dt\| \leq d\|x(t)\|/dt, \end{aligned}$$

almost everywhere in  $(t_1, t_0)$ . On the other hand,

$$\int_{t_1}^{t_0} \frac{dm_\eta(t)}{dt} dt > \int_{t_1}^{t_0} \frac{d\|x(t)\|}{dt} dt$$

if and only if

$$m_\eta(t_0) - m_\eta(t_1) = m_\eta(t_0) - \|x(t_1)\| > \|x(t_0)\| - \|x(t_1)\|.$$

Hence we get a contradiction. This proves Lemma 2.  $\square$

*Proof of Lemma 1.* Since  $u \in \text{Lip}(\Omega_T)$ , there exists  $L \geq 0$  such that

$$|u(t_1, x^1) - u(t_2, x^2)| \leq L(|t_1 - t_2| + \|x^1 - x^2\|),$$

$$\forall (t_1, x^1), (t_2, x^2) \in ([\theta, t_0] \times \mathbb{R}^n) \cap \Gamma(t_0, x_0).$$

By the absolute continuity of the Lebesgue integral, Lemma 1 will be proved if we can show that

$$(6) \quad |\varphi(t_1) - \varphi(t_2)| \leq L \left[ |t_1 - t_2| + (\|x_0\| + 1) \cdot \exp \int_{\theta}^{t_0} k(t) dt \cdot \int_{[t_1, t_2]} k(t) dt \right],$$

$$\forall t_1, t_2 \in [\theta, t_0].$$

Now let

$$\varphi(t_1) \geq \varphi(t_2) \quad \text{and} \quad \varphi(t_1) = |u(t_1, x(t_1))|,$$

for some  $x(\cdot) \in \Sigma_I(t_0, x_0)$ . Since  $x(t_2) \in Z(t_2, t_0, x_0)$ , we have

$$\begin{aligned} 0 \leq \varphi(t_1) - \varphi(t_2) &= |u(t_1, x(t_1))| - \varphi(t_2) \\ &\leq |u(t_1, x(t_1))| - |u(t_2, x(t_2))| \leq |u(t_1, x(t_1)) - u(t_2, x(t_2))| \\ &\leq L[|t_1 - t_2| + \|x(t_1) - x(t_2)\|] = L \left[ |t_1 - t_2| + \left\| \int_{[t_1, t_2]} \frac{dx}{dt}(t) dt \right\| \right] \\ &\leq L \left[ |t_1 - t_2| + \int_{[t_1, t_2]} k(t) \cdot (\|x(t)\| + 1) dt \right]. \end{aligned}$$

Therefore, (6) follows from Lemma 2. The proof is then complete.  $\square$

Going back to the proof of Theorem 1, we put now

$$f(t) = \int_0^t k(\tau) d\tau, \quad t \in [0, T].$$

By Lemma 2 and the definition of  $\varphi(\cdot)$ , the inequality (2.a) will be obtained if we show that

$$(7) \quad \varphi(t) \leq \varphi(0) \cdot \exp[C(x_0) \cdot f(t)], \quad \forall t \in [0, t_0].$$

For arbitrary  $\mu > 0$ , let

$$\psi(t) = \psi_\mu(t) = (\varphi(0) + \mu) \cdot \exp[(C(x_0) + \mu) \cdot (f(t) + \mu t)].$$

To get (7), we have only to prove that

$$(8) \quad \varphi(t) < \psi_\mu(t), \quad \forall t \in [0, t_0].$$

Let  $\omega = \psi - \varphi$ . Then (8) is equivalent to  $\omega(t) > 0, \forall t \in [0, t_0]$ . Of course,  $\omega(0) = \mu > 0$ . We shall show that  $\omega(t) \geq \omega(0), \forall t \in [0, t_0]$ . Assume this is false, so there exists  $t' \in (0, t_0)$  such that  $\omega(t') < \omega(0)$ .

It is well known that there exists  $G_1 \subset (0, T)$ ,  $\text{mes}(G_1) = 0$  such that

$$\frac{df}{dt}(t) = k(t), \quad \forall t \in (0, T) \setminus G_1.$$

By the hypothesis of Theorem 1, there exists  $G_2 \subset (0, T)$ ,  $\text{mes}(G_2) = 0$  such that  $\Omega_T \setminus (G_2 \times \mathbb{R}^n) \subset \text{Dif}(u)$ , and (1) holds for all  $t \in (0, T) \setminus G_2$ .

The absolute continuity of  $\omega(\cdot)$  on  $[\theta, t_0]$  implies

$$\text{mes}(\omega(G \cap [\theta, t_0])) = 0, \quad \forall \theta \in (0, t_0), \quad G = G_1 \cup G_2.$$

So

$$(9) \quad \text{mes}(\omega(G \cap [0, t_0])) = \lim_{\theta \searrow 0} \text{mes}(\omega(G \cap [\theta, t_0])) = 0.$$

From (9) and  $\omega(\cdot) \in C(I)$ , we could find a number  $\lambda$  with

$$\max\{0, \omega(t')\} < \lambda < \omega(0) \quad \text{and} \quad \lambda \in \omega[0, t'] \setminus \omega(G \cap [0, t_0]).$$

Let

$$t_* = \inf\{t \in [0, t'] \mid \omega(t) = \lambda\}.$$

It is obvious that  $\omega(t_*) = \lambda$ ,  $t_* \in (0, t') \setminus G$ , and that  $\omega(t) > \lambda$ ,  $\forall t \in [0, t_*)$ .

Suppose that

$$\varphi(t_*) = s \cdot u(t_*, x_*), \quad s = \text{sign } u(t_*, x_*),$$

for some  $x_* \in Z(t_*, t_0, x_0)$ ; i.e., there exists a function  $x^*(\cdot) \in \Sigma_I(t_0, x_0)$  so that  $x^*(t_*) = x_*$ . Choose  $l \in \mathbb{R}^n$  with

$$\langle l, s \cdot \nabla_x u(t_*, x_*) \rangle = -\|\nabla_x u(t_*, x_*)\|, \quad \|l\| = 1.$$

The system of differential equations

$$\frac{dy}{dp}(p) = (1 + \|y(p)\|) \cdot l$$

has a classical (i.e., continuously differentiable) solution on  $\mathbb{R}^1$  satisfying the condition  $y(f(t_*)) = x_*$ . Let  $x(t) = y(f(t))$ ,  $t \in [0, T]$ . Of course,  $x(\cdot)$  is absolutely continuous on  $[0, T]$ ,  $x(t_*) = x_*$ , and

$$(10) \quad \frac{dx}{dt}(t) = \frac{df}{dt}(t) \cdot \frac{dy}{dp}(f(t)) = k(t) \cdot (1 + \|x(t)\|) \cdot l, \quad \forall t \in (0, T) \setminus G_1.$$

The function  ${}_*x(\cdot)$  defined by

$${}_*x(t) = \begin{cases} x(t) & \text{if } 0 \leq t \leq t_*, \\ x^*(t) & \text{if } t_* \leq t \leq t_0, \end{cases}$$

belongs to  $\Sigma_I(t_0, x_0)$ . Hence,

$$x(t) \in Z(t, t_0, x_0), \quad \forall t \in [0, t_*].$$

This implies

$$(11) \quad s \cdot u(t, x(t)) \leq |u(t, x(t))| \leq \varphi(t) = \psi(t) - \omega(t) < \psi(t) - \lambda$$

for all  $t \in [0, t_*)$ . Besides that,

$$(12) \quad s \cdot u(t_*, x(t_*)) = |u(t_*, x_*)| = \varphi(t_*) = \psi(t_*) - \omega(t_*) = \psi(t_*) - \lambda.$$

Since  $t_* \in (0, T) \setminus G$ , we see that:

- (i)  $u$  is differentiable at  $(t_*, x_*)$ ,
- (ii)  $x(\cdot)$  is differentiable at  $t_*$  with  $dx/dt(t_*) = k(t_*) \cdot (1 + \|x_*\|) \cdot l$ ,
- (iii)  $\psi(\cdot)$  is differentiable at  $t_*$  with

$$\frac{d\psi}{dt}(t_*) = (C(x_0) + \mu) \cdot (k(t_*) + \mu) \cdot \psi(t_*).$$

So it follows from (11), (12) that

$$\frac{d}{dt}[s \cdot u(t, x(t))]|_{t=t_*} \geq \frac{d\psi}{dt}(t_*).$$

Consequently,

$$\begin{aligned} s \cdot \frac{\partial u}{\partial t}(t_*, x(t_*)) + \langle dx/dt(t_*), s \cdot \nabla_x u(t_*, x(t_*)) \rangle \\ \geq (C(x_0) + \mu) \cdot (k(t_*) + \mu) \cdot \psi(t_*). \end{aligned}$$

Hence,

$$\begin{aligned} s \cdot \frac{\partial u}{\partial t}(t_*, x_*) + k(t_*)(1 + \|x_*\|) \cdot \langle l, s \cdot \nabla_x u(t_*, x_*) \rangle \\ \geq (C(x_0) + \mu) \cdot (k(t_*) + \mu) \cdot (|u(t_*, x_*)| + \lambda). \end{aligned}$$

Because  $\mu > 0$  and  $\lambda > 0$ , the last inequality implies that

$$(13) \quad \left| \frac{\partial u}{\partial t}(t_*, x_*) \right| > k(t_*) \cdot [(1 + \|x_*\|) \cdot \|\nabla_x u(t_*, x_*)\| + C(x_0) \cdot |u(t_*, x_*)|].$$

Clearly, (13) contradicts (1), which shows that there could not exist any  $t' \in [0, t_0]$  with  $\omega(t') < \omega(0)$ . So

$$\omega(t) \geq \omega(0) > 0 \quad \text{for all } t \in [0, t_0];$$

therefore, (8) is proved. This completes the proof of Theorem 1.  $\square$

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