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ON THE DIMENSIONAL PROPERTIES OF THE STONE-ČECH REMAINDER OF P_0-SPACES

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ABSTRACT. A space X is called a P_0-space if there exists a perfect mapping f from X onto a metric space Y such that $\dim f = \sup\{f^{-1}(y): y \in Y\} = 0$. We prove that the P_0-space X is almost weakly infinite dimensional iff the remainder $\beta X \backslash X$ of the Stone-Čech compactification βX of X is A-weakly infinite dimensional. Furthermore we prove that $\Delta(\beta X \backslash X) = \operatorname{ind}(\beta X \backslash X) = \operatorname{Ind}(\beta X \backslash X) = \dim(\beta X \backslash X)$ for the P_0-space X.

0. INTRODUCTION

Unless stated otherwise, all spaces under consideration are normal. Our terminology follows [1, 7]. A closed continuous mapping $f: X \to Y$ is perfect if the inverse image $f^{-1}(y)$ of any point $y \in Y$ is compact. A Tychonoff space X is a paracompact P-space if it can be mapped perfectly onto a metrizable space [3]. A subclass of the class of P-spaces, namely, P_0-spaces, is introduced and investigated in [5, 11]. A space X is called a P_0-space if there exists a perfect mapping f from X onto a metric space Y such that $\dim f = \sup\{\dim f^{-1}(y): y \in Y\} = 0$. A mapping $f: X \to Y$ is called n-multiple if $|f^{-1}(y)| \leq n$ for every $y \in Y$. The set C is a partition between sets A and B in the space X if there exist open disjoint sets V and W satisfying the conditions $A \subset V$, $B \subset W$, and $X \backslash C = V \cup W$. A space X is said to be weakly infinite dimensional or A-weakly infinite dimensional if for every sequence $\{(A_i, B_i): i \in N\}$ of pairs of disjoint closed subsets of X, there exists a sequence $\{C_i: i \in N\}$ such that $\bigcap\{C_i: i \in N\} = \varnothing$, where C_i is a partition between A_i and B_i in X for all $i \in N$.

1. AUXILIARY ASSERTIONS

Theorem 1.1. *Consider a perfect mapping $f: X \to Y$ from a Tychonoff space X onto a space Y such that the set $F = \operatorname{Cl}(\{y \in Y: |f^{-1}(y)| > n\})$ is compact. Then the mapping $\tilde{f}|(\beta X \backslash X)$ is n-multiple, where $\tilde{f}: \beta X \to \beta Y$ is the continuous extension of the mapping f on Stone-Čech compactifications of the spaces X and Y.*

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Proof. Let $y_0 \in \beta Y \backslash Y$ and assume $|\tilde{f}^{-1}(y_0)| > n$. Consider distinct points $x_0, x_1, \dots, x_n \in \tilde{f}^{-1}(y_0)$. There exists a continuous mapping $h: \beta X \to [-1, 3n]$, where $h^{-1}(-1) \supseteq f^{-1}F$ and $h(x_k) = 3k$ for all $k \leq n$. Put $\Phi_i = h^{-1}[3i-1, 3i+1]$. We shall prove that there exist closed sets $H_0, H_1, \dots, H_n$ in X such that $H_i \subset \Phi_i$, $x_i \in \mathrm{Cl}_{\beta X} H_i$, and $fH_i \cap fH_j = \varnothing$, for all $i \neq j$ and $i, j \leq n$. We have $\bigcap\{f(X \cap \Phi_i): i \leq n\} = \varnothing$.

Suppose there exists a subset $L \subset \{0, 1, 2, \dots, n\}$ such that $|L| \geq 2$, $\Phi_L = \bigcap\{f(X \cap \Phi_i): i \in L\} \neq \varnothing$, and $\Phi_L \cap \tilde{f}\Phi_j = \varnothing$ for all $j \notin L$. If $j \notin L$, then $y_0 \in \mathrm{Cl}_{\beta X} f(\Phi_j \cap X)$ and $\mathrm{Cl}_{\beta Y} \Phi_L \cap \mathrm{Cl}_{\beta Y} f(\Phi_j \cap X) = \varnothing$. Thus $y_0 \notin \mathrm{Cl}_{\beta Y} \Phi_L$, and there exists an open set V in βY such that $y_0 \in V$ and $\mathrm{Cl}_{\beta Y} V \cap \mathrm{Cl}_{\beta Y} \Phi_L = \varnothing$. Put $\Phi_i^1 = \Phi_i \cap \mathrm{Cl}_{\beta X} \tilde{f}^{-1}(V)$. Then $\bigcap\{f(X \cap \Phi_i^1): i \in L\} = \varnothing$ and $x_j \in \mathrm{Cl}_{\beta X}(\Phi_j^1 \cap X)$ for all $i \leq n$. Repeat this process a finite number of times to construct closed sets $\{\Phi_i^k : i \leq n\}$ in βX such that $\Phi_i^k \subset \Phi_i^{k-1} \subset \Phi_i$, $x_i \in \mathrm{Cl}_{\beta X}(\Phi_i^k \cap X)$, and $f(\Phi_i^k \cap X) \cap f(\Phi_j^k \cap X) = \varnothing$ for all $i, j \leq n$ and $i \neq j$. Then the sets $\{H_i = f(\Phi_i^k \cap X): i \leq n\}$ are closed, disjoint in Y and $y_0 \in \bigcap\{\mathrm{Cl}_{\beta Y} H_i: i \leq n\}$. This fact contradicts the normality of the space Y. The proof is therefore completed. The case $F = \varnothing$ is studied in [10].

The mapping $f: X \to Y$ is called locally multifinite if for every point $x \in X$ there exists a neighbourhood O_x of the point x and a number $n \in N$ such that $\sup\{|f^{-1}(y)|: y \in O_x\} = n$.

Proposition 1.2. *Consider a perfect mapping $f: X \to Y$ from a space X onto a space Y such that the set $F \subset Y$ is compact and for every open set U in Y with $F \subset U$, there exists a number $n \in N$ such that $|f^{-1}(y)| \leq n$ for all $y \in Y \backslash U$. Then the mapping $\tilde{f}|(\beta X \backslash f^{-1}F)$ is locally multifinite, where $\tilde{f}: \beta X \to \beta Y$ is the continuous extension of the mapping f on Stone-Čech compactifications of X and Y.*

Proof. Consider a point $y_0 \in \beta Y \backslash F$. There exist a closed set $\Phi \subset Y$ and a number $n \in N$ such that $y_0 \in \mathrm{Cl}_{\beta Y} \Phi$, $F \subset \mathrm{int}\,\Phi$, and $|f^{-1}(y)| \leq n$ for all $y \in Y \backslash \Phi$. Put $Y_1 = Y \cup \mathrm{Cl}_{\beta X} \Phi$, $X_1 = \tilde{f}^{-1} Y_1$, and $g = \tilde{f}|X_i$. The space Y_1 is normal, $\beta Y_1 = \beta Y$, and $\beta X_1 = \beta X$. Theorem 1.1 completes the proof.

Proposition 1.3. *If $\{X_n: n \in N\}$ are closed subspaces of the P_0-space X, then there exist a metric space Z and a perfect mapping $f: X \to Z$ onto Z such that $\dim f = 0$ and $\dim fX_n \leq \dim X_n$ for all $n \in N$.*

Proof. Consider a perfect mapping $g: X \to Z$ onto a metric space Z such that $\dim g = 0$. By using Arhangelskii's factorization theorem [2] there exist a metric space Y and continuous mappings $f: X \to Y$, $h: Y \to Z$ such that $g = h \circ f$ and $\dim fX_n \leq \dim X_n$ for all $n \in N$. Clearly $\dim f \leq \dim g \leq 0$. Using the method introduced in [5] we construct the mapping f, which will be perfect.

Proposition 1.4 [11]. *Let $f: X \to Y$ and $g: Z \to Y$ be perfect mappings. Put $P = \Gamma(X, Y, Z, f, g) = \{\{x\} \times g^{-1}(f(x)): x \in X\} \subset X \times Z$, $\tilde{f}: P \to Z$, and $\tilde{g}: P \to X$ such that $\tilde{f}(x, z) = z$ and $\tilde{g}(x, z) = x$ for all $(x, z) \in P$. Then*

(1) *The mappings $\tilde{f}$ and $\tilde{g}$ are perfect;*
(2) *If $\dim f = 0$ and $\dim Z = 0$, then $\dim P = 0$;*
(3) *$|g^{-1}(f(x))| = |\tilde{g}^{-1}(x)|$ for all $x \in X$.*

Proposition 1.5. *If $\{X_n : n \in N\}$ are finite-dimensional closed subspaces of the P_0-space X, then there exist a space Z and perfect mapping $h: Z \to X$ onto X such that*

(1) $\dim Z = 0$ *and the space Z is paracompact*;
(2) $|h^{-1}(x)| \leq 1 + \dim X_n$ *for all $x \in X_n$ and $n \in N$.*

Proof. Using Proposition 1.3 there exist a metric space Y and a perfect mapping $f: X \to Y$ such that $\dim f = 0$ and $\dim fX_n \leq \dim X_n$ for all $n \in N$. By Arhangelskii's factorization theorem [4] there exist a metric space S such that $\dim S = 0$ and an open compact mapping $\psi: S \to Y$ onto the space Y. Nedev [9] proved that there exists a closed subspace $S_1 \subset S$ such that $\psi S_1 = Y$, the mapping $g = \psi|S_1$ is perfect, and $|g^{-1}(y)| \leq 1 + \dim fx_n$ for all $y \in fX_n$. Put $Z = \Gamma(X, Y, S_1, f, g)$ and $h = \tilde{g}$, where $\tilde{g}: Z \to Y$ is defined by $\tilde{g}(x, z) = z$ for all $(x, z) \in Z$. Proposition 1.4 completes the proof.

A space X is almost n-dimensional if there exists a compact set $C \subseteq X$ such that $\dim Y \leq n$ for every closed set Y in X contained in $X \backslash C$.

From the results of the work [6], it follows that

Theorem 1.6. *If the P_0-space X is almost n-dimensional, then there exists a compact G_δ-set F of X such that* $\dim(X \backslash F) = n$.

2. Fundamental results

For a Tychonoff space X we define the dimension function ΔX as follows: $\Delta X \leq n$ if there exist a space Z such that $\dim Z = 0$ and a $(n+1)$-multiple closed continuous mapping $f: Z \to X$ onto the space X. The considered invariant ΔX is introduced first by Ponamarev [12] by using directed families of closed covers. The above definition of the dimension function ΔX is equivalent to the definition of Ponamarev, which is introduced by Pasynkov [10].

Theorem 2.1. *For every P_0-space X the following equalities hold*:

$$\dim(\beta X \backslash X) = \operatorname{Ind}(\beta X \backslash X) = \operatorname{ind}(\beta X \backslash X) = \Delta(\beta X \backslash X).$$

Proof. Put $X^* = \beta X \backslash X$. The space X^* is Lindelöf [8] since every compact subset of X is contained in a compact subset of countable character (i.e., the space X is of countable type). Hence $\dim X^* \leq \operatorname{ind} X^* \leq \operatorname{Ind} X^* \leq \Delta X^*$. If $\dim X^* = n$, then there exists a compact G_δ-set $F \subset X$ such that $\dim(X \backslash F) = n$ [6]. Thus $X \backslash F = \bigcup\{X_m : m \in N\}$, where the sets X_m are closed in X and $\dim X_m \leq n$. By Proposition 1.5 there exist a space Z and a perfect mapping $h: Z \to X$ onto X such that $\dim Z = 0$ and $|h^{-1}(x)| \leq n+1$ for all $x \in X \backslash F$.

Consider a continuous extension mapping $\beta h: \beta Z \to X$ of the mapping h. The set $\operatorname{Cl}(\{x \in X : |h^{-1}(x)| \geq n+2\})$ is compact and contained in F. By Theorem 1.1 it follows that $|\beta h^{-1}(x)| \leq n+1$ for all $x \in X^*$. The space $Z^* = \beta Z \backslash Z$ is Lindelöf. Thus $\dim Z^* \leq \dim \beta Z = \dim Z = 0$. The mapping $g = h|Z^* : Z^* \to X^*$ is closed and $|g^{-1}(x)| \leq n+1$. Thus $\Delta X^* \leq n$. Hence $n \leq \dim X^* \leq \operatorname{ind} X^* \leq \operatorname{Ind} X^* \leq \Delta X^* \leq n$. The proof is complete.

For every space X consider the notations

$$R_n(X) = \{x : \operatorname{loc\,dim}_x X \geq n\} \quad \text{and} \quad R_\infty(X) = \bigcap\{R_n(X) : n \in N\}.$$

For a space X we define the following local dimensions:

(1) $\operatorname{loc}\Delta X < \infty$ if there exist a space Z such that $\dim Z = 0$ and a closed locally multifinite mapping f from Z onto X.
(2) $\operatorname{Loc}\Delta X < \infty$ if there exist a space Z such that $\dim Z = 0$ and a closed finite-dimensional mapping f from Z onto X.

Clearly $\operatorname{Loc}\Delta X \leq \operatorname{loc}\Delta X$ and $\operatorname{loc}\dim X \leq \operatorname{loc}\Delta X$ for every space X. If $\operatorname{Loc}\Delta X < \infty$ and the space X is paracompact, then X is A-weakly infinite dimensional.

A space X is almost weakly infinite dimensional if there exists a compact set $F \subset X$ such that $\dim(X \backslash U) < \infty$ for every open set U in X containing F.

Theorem 2.2. *For the P_0-space X the following statements are equivalent*:

(1) *The space X is almost weakly infinite dimensional.*
(2) *The set $R_\infty(X)$ is compact, and for every open set U in X with $R_\infty(X) \subset U$, there exists $n \in N$ such that $R_n(X) \subset U$.*
(3) $\dim \Phi < \infty$ *for every compact set $\Phi \subset \beta X \backslash X$.*
(4) $\Delta\Phi < \infty$ *for every compact set $\Phi \subset \beta X \backslash X$.*
(5) *Every compact set $\Phi \subset \beta X \backslash X$ is weakly infinite dimensional.*
(6) *The space $\beta X \backslash X$ is A-weakly infinite dimensional.*
(7) $\operatorname{loc}\Delta(\beta X \backslash X) < \infty$.
(8) $\operatorname{loc}\dim(\beta X \backslash X) < \infty$.
(9) $\operatorname{Loc}\Delta(\beta X \backslash X) < \infty$.

Proof. Implications (1) $\to$ (2) $\to$ (3) $\to$ (5) $\to$ (1) follow from Theorem 3.6 in [6]. Implications (9) $\to$ (6) $\to$ (5), (7) $\to$ (9), and (7) $\to$ (8) $\to$ (6) are obvious. It remains to prove that (2) $\to$ (7). Using Arhangelskii's factorization theorem and the theorem of Nedev as in the proof of Proposition 1.5, we construct a metric space Z such that $\dim Z = 0$ and a perfect mapping f from Z onto X such that $R_n(X) \supset \{x \in X : |f^{-1}(x)| \geq n + 1\}$. Proposition 1.2 completes the proof.

A space X is almost completely n-dimensional if the set $R_n(X)$ is not countably compact and there exists a compact set $F \subset X$ such that $\dim(X \backslash F) = \Delta(X \backslash F) = n$.

By the proof of Theorem 2.1 we establish the following fact.

Theorem 2.3. *For the P_0-space X the following statements are equivalent*:

(1) *The space X is almost n-dimensional.*
(2) *The space X is almost completely n-dimensional.*

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