ON THE DIMENSIONAL PROPERTIES
OF THE STONE-ČECH REMAINDER OF $P_0$-SPACES

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Abstract. A space $X$ is called a $P_0$-space if there exists a perfect mapping $f$ from $X$ onto a metric space $Y$ such that $\dim f = \sup \{f^{-1}(y) : y \in Y\} = 0$. We prove that the $P_0$-space $X$ is almost weakly infinite dimensional iff the remainder $\beta X \setminus X$ of the Stone-Cech compactification $\beta X$ of $X$ is $\omega$-weakly infinite dimensional. Furthermore we prove that $\Delta(\beta X \setminus X) = \text{ind}(\beta X \setminus X) = \text{Ind}(\beta X \setminus X) = \dim(\beta X \setminus X)$ for the $P_0$-space $X$.

0. Introduction

Unless stated otherwise, all spaces under consideration are normal. Our terminology follows [1, 7]. A closed continuous mapping $f: X \to Y$ is perfect if the inverse image $f^{-1}(y)$ of any point $y \in Y$ is compact. A Tychonoff space $X$ is a paracompact $P$-space if it can be mapped perfectly onto a metrizable space [3]. A subclass of the class of $P$-spaces, namely, $P_0$-spaces, is introduced and investigated in [5, 11]. A space $X$ is called a $P_0$-space if there exists a perfect mapping $f$ from $X$ onto a metric space $Y$ such that $\dim f = \sup \{\dim f^{-1}(y) : y \in Y\} = 0$. A mapping $f: X \to Y$ is called $n$-multiple if $|f^{-1}(y)| \leq n$ for every $y \in Y$. The set $C$ is a partition between sets $A$ and $B$ in the space $X$ if there exist open disjoint sets $V$ and $W$ satisfying the conditions $A \subseteq V$, $B \subseteq W$, and $X \setminus C = V \cup W$. A space $X$ is said to be weakly infinite dimensional or $\omega$-weakly infinite dimensional if for every sequence $\{(A_i, B_i) : i \in \mathbb{N}\}$ of pairs of disjoint closed subsets of $X$, there exists a sequence $\{C_i : i \in \mathbb{N}\}$ such that $\bigcap \{C_i : i \in \mathbb{N}\} = \emptyset$, where $C_i$ is a partition between $A_i$ and $B_i$ in $X$ for all $i \in \mathbb{N}$.

1. Auxiliary assertions

Theorem 1.1. Consider a perfect mapping $f: X \to Y$ from a Tychonoff space $X$ onto a space $Y$ such that the set $F = \text{Cl}(\{y \in Y : |f^{-1}(y)| > n\})$ is compact. Then the mapping $\bar{f}(\beta X \setminus X)$ is $n$-multiple, where $\bar{f}: \beta X \to \beta Y$ is the continuous extension of the mapping $f$ on Stone-Čech compactifications of the spaces $X$ and $Y$.

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Proof. Let $y_0 \in \beta Y \setminus Y$ and assume $|\hat{f}^{-1}(y_0)| > n$. Consider distinct points $x_0, x_1, \ldots, x_n \in \hat{f}^{-1}(y_0)$. There exists a continuous mapping $h: \beta X \to [-1, 3n]$, where $h^{-1}(-1) \supseteq f^{-1} F$ and $h(x_k) = 3k$ for all $k \leq n$. Put $\Phi_i = h^{-1}[3i-1, 3i+1]$. We shall prove that there exist closed sets $H_0, H_1, \ldots, H_n$ in $X$ such that $H_i \subseteq \Phi_i$, $x_i \in \text{Cl}_{\beta X} H_i$, and $f H_i \cap f H_j = \emptyset$, for all $i \neq j$ and $i, j \leq n$. We have $\bigcap \{f (X \cap \Phi_i): i \leq n\} = \emptyset$.

Suppose there exists a subset $L \subseteq \{0, 1, 2, \ldots, n\}$ such that $|L| > 2$, $\Phi_L = \bigcap \{f (X \cap \Phi_i): i \in L\} \neq \emptyset$, and $\Phi_L \cap \hat{f} \Phi_i = \emptyset$ for all $j \notin L$. If $j \notin L$, then $y_0 \in \text{Cl}_{\beta X} f (\Phi_j \cap X)$ and $\text{Cl}_{\beta Y} \Phi_L \cap \text{Cl}_{\beta Y} f (\Phi_j \cap X) = \emptyset$. Thus $y_0 \notin \text{Cl}_{\beta Y} \Phi_L$, and there exists an open set $V$ in $\beta Y$ such that $y_0 \in V$ and $\text{Cl}_{\beta Y} V \cap \text{Cl}_{\beta Y} \Phi_L = \emptyset$. Put $\Phi_i^1 = \Phi_i \cap \text{Cl}_{\beta X} \hat{f}^{-1}(V)$. Then $\bigcap \{f (X \cap \Phi_i^1): i \in L\} = \emptyset$ and $x_j \in \text{Cl}_{\beta X} (\Phi_j \cap X)$ for all $i \leq n$. Repeat this process a finite number of times to construct closed sets $\{\Phi_i^k: i \leq n\}$ in $\beta X$ such that $\Phi_i^k \subseteq \Phi_i^{k-1} \subseteq \Phi_i$, $x_i \in \text{Cl}_{\beta X} (\Phi_i \cap X)$, and $f (\Phi_i^k \cap X) \cap f (\Phi_j^k \cap X) = \emptyset$ for all $i, j \leq n$ and $i \neq j$. Then the sets $\{H_i = f (\Phi_i \cap X): i \leq n\}$ are closed, disjoint in $Y$ and $y_0 \in \bigcap \{\text{Cl}_{\beta Y} H_i: i \leq n\}$. This fact contradicts the normality of the space $Y$. The proof is therefore completed. The case $F = \emptyset$ is studied in [10].

The mapping $f: X \to Y$ is called locally multifinite if for every point $x \in X$ there exists a neighbourhood $O_x$ of the point $x$ and a number $n \in N$ such that $\sup \{|f(x)(y)|: y \in O_x\} = n$.

Proposition 1.2. Consider a perfect mapping $f: X \to Y$ from a space $X$ onto a space $Y$ such that the set $F \subseteq Y$ is compact and for every open set $U$ in $Y$ with $F \subseteq U$, there exists a number $n \in N$ such that $|f^{-1}(y)| \leq n$ for all $y \in Y \setminus U$. Then the mapping $f (\beta X \setminus f^{-1} F)$ is locally multifinite, where $\hat{f}: \beta X \to \beta Y$ is the continuous extension of the mapping $f$ on Stone-Cech compactifications of $X$ and $Y$.

Proof. Consider a point $y_0 \in \beta Y \setminus F$. There exist a closed set $\Phi \subseteq Y$ and a number $n \in N$ such that $y_0 \in \text{Cl}_{\beta Y} \Phi$, $F \subseteq \text{int} \Phi$, and $|f^{-1}(y)| \leq n$ for all $y \in Y \setminus \Phi$. Put $Y_1 = Y \cup \text{Cl}_{\beta X} \Phi$, $X_1 = \hat{f}^{-1} Y_1$, and $g = \hat{f} |X_1$. The space $Y_1$ is normal, $\beta Y_1 = \beta Y$, and $\beta X_1 = \beta X$. Theorem 1.1 completes the proof.

Proposition 1.3. If $\{X_n: n \in N\}$ are closed subspaces of the $P_0$-space $X$, then there exist a metric space $Z$ and a perfect mapping $f: X \to Z$ onto $Z$ such that $\dim f = 0$ and $\dim f X_n \leq \dim X_n$ for all $n \in N$.

Proof. Consider a perfect mapping $g: X \to Z$ onto a metric space $Z$ such that $\dim g = 0$. By using Arhangelskii's factorization theorem [2] there exist a metric space $Y$ and continuous mappings $f: X \to Y$, $h: Y \to Z$ such that $g = h \circ f$ and $\dim f X_n \leq \dim X_n$ for all $n \in N$. Clearly $\dim f \leq \dim g \leq 0$. Using the method introduced in [5] we construct the mapping $f$, which will be perfect.

Proposition 1.4 [11]. Let $f: X \to Y$ and $g: Z \to Y$ be perfect mappings. Put $P = \Gamma(X, Y, Z, f, g) = \{(x) \times g^{-1}(f(x)): x \in X\} \subseteq X \times Z$, $\hat{f}: P \to Z$, and $\hat{g}: P \to X$ such that $\hat{f}(x, z) = z$ and $\hat{g}(x, z) = x$ for all $(x, z) \in P$. Then

1. The mappings $\hat{f}$ and $\hat{g}$ are perfect;
2. If $\dim f = 0$ and $\dim Z = 0$, then $\dim P = 0$;
3. $|g^{-1}(f(x))| = |\hat{g}^{-1}(x)|$ for all $x \in X$. 
Proposition 1.5. If \( \{X_n: n \in \mathbb{N}\} \) are finite-dimensional closed subspaces of the \( P_0 \)-space \( X \), then there exist a space \( Z \) and perfect mapping \( h: Z \to X \) onto \( X \) such that

1. \( \dim Z = 0 \) and the space \( Z \) is paracompact;
2. \( |h^{-1}(x)| \leq 1 + \dim X_n \) for all \( x \in X_n \) and \( n \in \mathbb{N} \).

**Proof.** Using Proposition 1.3 there exist a metric space \( Y \) and a perfect mapping \( f: X \to Y \) such that \( \dim f = 0 \) and \( \dim fX_n \leq \dim X_n \) for all \( n \in \mathbb{N} \). By Arhangel'skii's factorization theorem [4] there exist a metric space \( S \) such that \( \dim S = 0 \) and an open compact mapping \( \psi: S \to Y \) onto the space \( Y \). Nedev [9] proved that there exists a closed subspace \( S_1 \subset S \) such that \( \psi S_1 = Y \), the mapping \( g = \psi|S_1 \) is perfect, and \( |g^{-1}(y)| \leq 1 + \dim fX_n \) for all \( y \in fX_n \). Put \( Z = \Gamma(X, Y, S_1, f, g) \) and \( h = g \), where \( g: Z \to Y \) is defined by \( g(x, z) = z \) for all \( (x, z) \in Z \). Proposition 1.4 completes the proof.

A space \( X \) is almost \( n \)-dimensional if there exists a compact set \( C \subset X \) such that \( \dim Y \leq n \) for every closed set \( Y \) in \( X \) contained in \( X \setminus C \).

From the results of the work [6], it follows that

**Theorem 1.6.** If the \( P_0 \)-space \( X \) is almost \( n \)-dimensional, then there exists a compact \( G_\delta \)-set \( F \) of \( X \) such that \( \dim (X \setminus F) = n \).

2. **Fundamental results**

For a Tychonoff space \( X \) we define the dimension function \( \Delta X \) as follows: \( \Delta X \leq n \) if there exist a space \( Z \) such that \( \dim Z = 0 \) and a \((n + 1)\)-multiple closed continuous mapping \( f: Z \to X \) onto the space \( X \). The considered invariant \( \Delta X \) is introduced first by Ponamarev [12] by using directed families of closed covers. The above definition of the dimension function \( \Delta X \) is equivalent to the definition of Ponamarev, which is introduced by Pasynkov [10].

**Theorem 2.1.** For every \( P_0 \)-space \( X \) the following equalities hold:

\[
\dim(\beta X \setminus X) = \text{Ind}(\beta X \setminus X) = \text{ind}(\beta X \setminus X) = \Delta(\beta X \setminus X).
\]

**Proof.** Put \( X^* = \beta X \setminus X \). The space \( X^* \) is Lindelöf [8] since every compact subset of \( X \) is contained in a compact subset of countable character (i.e., the space \( X \) is of countable type). Hence \( \dim X^* \leq \text{ind} X^* \leq \text{Ind} X^* \leq \Delta X^* \). If \( \dim X^* = n \), then there exists a compact \( G_\delta \)-set \( F \subset X \) such that \( \dim (X \setminus F) = n \) [6]. Thus \( X \setminus F = \bigcup \{X_m: m \in \mathbb{N}\} \), where the sets \( X_m \) are closed in \( X \) and \( \dim X_m \leq n \). By Proposition 1.5 there exist a space \( Z \) and a perfect mapping \( h: Z \to X \) such that \( \dim Z = 0 \) and \( |h^{-1}(x)| \leq n + 1 \) for all \( x \in X \setminus F \).

Consider a continuous extension mapping \( \beta h: \beta Z \to X \) of the mapping \( h \). The set \( Cl(\{x \in X: |h_1(x)| \geq n + 2\}) \) is compact and contained in \( F \). By Theorem 1.1 it follows that \( |\beta h_1(x)| \leq n + 1 \) for all \( x \in X^* \). The space \( Z^* = \beta Z \setminus Z \) is Lindelöf. Thus \( \dim Z^* \leq \dim \beta Z = \dim Z = 0 \). The mapping \( g = h|Z^*: Z^* \to X^* \) is closed and \( |g^{-1}(x)| \leq n + 1 \). Thus \( \Delta X^* \leq n \). Hence \( n \leq \dim X^* \leq \text{ind} X^* \leq \text{Ind} X^* \leq \Delta X^* \leq n \). The proof is complete.

For every space \( X \) consider the notations

\[
R_n(X) = \{x: \text{loc dim}_x X \geq n\} \quad \text{and} \quad R_\infty(X) = \bigcap \{R_n(X): n \in \mathbb{N}\}.
\]
For a space $X$ we define the following local dimensions:

1. $\text{loc} \Delta X < \infty$ if there exist a space $Z$ such that $\dim Z = 0$ and a closed locally multifinite mapping $f$ from $Z$ onto $X$.
2. $\text{Loc} \Delta X < \infty$ if there exist a space $Z$ such that $\dim Z = 0$ and a closed finite-dimensional mapping $f$ from $Z$ onto $X$.

Clearly $\text{Loc} \Delta X \leq \text{loc} \Delta X$ and $\text{loc} \dim X \leq \text{loc} \Delta X$ for every space $X$. If $\text{Loc} \Delta X < \infty$ and the space $X$ is paracompact, then $X$ is $A$-weakly infinite dimensional.

A space $X$ is almost weakly infinite dimensional if there exists a compact set $F \subset X$ such that $\dim(X \setminus U) < \infty$ for every open set $U$ in $X$ containing $F$.

**Theorem 2.2.** For the $P_0$-space $X$ the following statements are equivalent:

1. The space $X$ is almost weakly infinite dimensional.
2. The set $R_\infty(X)$ is compact, and for every open set $U$ in $X$ with $R_\infty(X) \subset U$, there exists $n \in \mathbb{N}$ such that $R_n(X) \subset U$.
3. $\dim \Phi < \infty$ for every compact set $\Phi \subset \beta X \setminus X$.
4. $\Delta \Phi < \infty$ for every compact set $\Phi \subset \beta X \setminus X$.
5. Every compact set $\Phi \subset \beta X \setminus X$ is weakly infinite dimensional.
6. The space $\beta X \setminus X$ is $A$-weakly infinite dimensional.
7. $\text{loc} \Delta(\beta X \setminus X) < \infty$.
8. $\text{loc} \dim(\beta X \setminus X) < \infty$.
9. $\text{Loc} \Delta(\beta X \setminus X) < \infty$.

**Proof.** Implications $(1) \rightarrow (2) \rightarrow (3) \rightarrow (5) \rightarrow (1)$ follow from Theorem 3.6 in [6]. Implications $(9) \rightarrow (6) \rightarrow (5)$, $(7) \rightarrow (9)$, and $(7) \rightarrow (8) \rightarrow (6)$ are obvious. It remains to prove that $(2) \rightarrow (7)$. Using Arhangel'skii's factorization theorem and the theorem of Nedev as in the proof of Proposition 1.5, we construct a metric space $Z$ such that $\dim Z = 0$ and a perfect mapping $f$ from $Z$ onto $X$ such that $R_n(X) \supset \{x \in X: |f^{-1}(x)| \geq n + 1\}$. Proposition 1.2 completes the proof.

A space $X$ is almost completely $n$-dimensional if the set $R_n(X)$ is not countably compact and there exists a compact set $F \subset X$ such that $\dim(X \setminus F) = \Delta(X \setminus F) = n$.

By the proof of Theorem 2.1 we establish the following fact.

**Theorem 2.3.** For the $P_0$-space $X$ the following statements are equivalent:

1. The space $X$ is almost $n$-dimensional.
2. The space $X$ is almost completely $n$-dimensional.

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