GENERALIZATIONS OF DEODHAR'S $\alpha$-LOCALIZATION FUNCTOR

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Abstract. In this paper we generalize the result of Deodhar (see Invent. Math. 57 (1980), 101-118) on $\alpha$-localization functors. Namely, we show that localization with respect to a larger family of left denominator sets "intertwines" with tensoring by finite-dimensional representations. In the language of the author's previous work, localization with respect to such a left denominator set produces a new example of an $\mathfrak{F}$-functor and an $\mathfrak{F}$-category.

1. Introduction

Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. In [D], Deodhar discovered that Enright's completion functor (see [E1, E2]) is actually a subfunctor of an $\alpha$-localization functor. This latter functor is obtained by noncommutative localization with respect to the left denominator set $S_\alpha = \{y^m_n | n \in N\}$ where $y_\alpha$ is a nilpotent element in $\mathfrak{n}_-$. Localization with respect to this set "intertwines" with tensoring by finite-dimensional $\mathfrak{g}$-modules. Below we find new examples of noncommutative localization which also intertwine with tensoring by finite-dimensional representations.

2. Preliminaries and notation

2.1. Recall that an additive category $\mathfrak{A}$ is a category satisfying the following three axioms:
   
   (i) $\mathfrak{A}$ has a zero object;
   
   (ii) any two objects in $\mathfrak{A}$ have a product; and
   
   (iii) for all objects $A, B \in \text{Ob}\mathfrak{A}$ the set of morphisms $\text{Hom}_\mathfrak{A}(A, B)$ forms an abelian group such that the composition

   $$\text{Hom}_\mathfrak{A}(A, B) \times \text{Hom}_\mathfrak{A}(B, C) \to \text{Hom}_\mathfrak{A}(A, C)$$

   is bilinear. One also has the following proposition.

   **Proposition** [HS, Chapter 2]. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two additive categories and $F : \mathfrak{A} \to \mathfrak{B}$ a functor. Then the following are equivalent:

   (i) $F$ preserves sums (of two objects).
   
   (ii) $F$ preserves products (of two objects).
(iii) For each $A, A' \in \text{Ob} \mathcal{A}$ one has that

$$F : \text{Hom}_{\mathcal{B}}(A, A') \to \text{Hom}_{\mathcal{B}}(FA, FA')$$

is a group homomorphism.

A functor satisfying the above equivalent conditions is called an additive functor.

For a vector space $V$ over a field $k$, let $T^n(V)$ denote the $n$-fold tensor product of $V$ with itself, and let $T^0 = k$. Then $T(V) := \bigoplus_{n=0}^{\infty} T^n(V)$ is the tensor algebra of $V$. Elements in $T^n(V)$ are called homogeneous of degree $n$. If $\mathfrak{g}$ is a Lie algebra, let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$.

3. $\mathcal{C}$-Categories and $\mathcal{C}$-functors

3.1. For any Lie algebra $\mathfrak{g}$ (possibly infinite dimensional) defined over a field $k$ of characteristic zero, let $M_{\mathfrak{g}}$ denote the category of all $\mathfrak{g}$-modules. Throughout we will assume that $\mathcal{C}$ denotes an additive subcategory of $M_{\mathfrak{g}}$ satisfying the following two conditions:

1. $\mathcal{C}$ is closed under tensoring; i.e., if $E_j, F_j \in \mathcal{C}$ and $f_j \in \text{Hom}_{\mathcal{B}}(E_j, F_j)$ for $j = 1, 2$, then $E_1 \otimes E_2, F_1 \otimes F_2 \in \text{Ob} \mathcal{C}$ and $f_1 \otimes f_2 \in \text{Hom}_{\mathcal{B}}(E_1 \otimes E_2, F_1 \otimes F_2)$.

2. $0 \in \text{Ob} \mathcal{C}$ as a $\mathfrak{g}$-module under the adjoint action.

For $F \in \text{Ob} \mathcal{C}$, $\mathfrak{g}$-modules $A$ and $B$, and $h \in \text{Hom}_{\mathcal{B}}(A, B)$, let $T_F$ denote the tensor product functor on $M_{\mathfrak{g}}$ given by $A \mapsto F \otimes A$ and $h \mapsto 1_F \otimes h$.

If $n \subset \mathfrak{g}$ is a Lie subalgebra and $\mathcal{C}$ is a subcategory of $M_{\mathfrak{g}}$, we shall use the symbol $T_F$ to denote the tensor product functor on $\mathcal{C}$ when no confusion is likely to arise. We call the category $\mathcal{C}$ an $\mathcal{C}$-category if it is additive and $T_F$ carries $\mathcal{C}$ into itself for all $F \in \text{Ob} \mathcal{C}$.

Now let $a$ and $b$ be two Lie subalgebras of $\mathfrak{g}$, and let $\mathcal{A}$ (resp. $\mathcal{B}$) be an additive subcategory of $M_{\mathfrak{g}}$ (resp. $M_{\mathfrak{g}}$). Suppose further that both $\mathcal{A}$ and $\mathcal{B}$ are $\mathfrak{g}$-categories and $\tau$ is a functor from $\mathcal{A}$ to $\mathcal{B}$. We call $\tau$ an intertwining functor (or $\mathfrak{g}$-intertwining functor when more precision is necessary) if $\tau$ is additive and there exists a natural equivalence, for each $F \in \text{Ob} \mathcal{C}$, $i_F : T_F \circ \tau \to \tau \circ T_F$.

Suppose $\tau$ is an intertwining functor, and let $\mathcal{J} = \{ i_F | F \in \text{Ob} \mathcal{C} \}$ denote the family of natural equivalences above. Recall that a natural equivalence $i_E : T_E \circ \tau \to \tau \circ T_E$ is a rule that assigns to each object $A$ of $\mathcal{A}$ an isomorphism $i_E(A) : T_E \circ \tau(A) \to \tau \circ T_E(A)$ such that for every homomorphism $f : A \to B$ in $\mathcal{A}$ one has $i_E(B) \circ ((T_E \circ \tau)(f)) = (\tau \circ T_E)(f) \circ i_E(A)$. For convenience we set $i_{E,A} = i_E(A)$ for all $A \in \text{Ob} \mathcal{A}$, $E \in \text{Ob} \mathcal{C}$.

Suppose now that for every $E, F \in \text{Ob} \mathcal{C}$ and $h \in \text{Hom}_{\mathcal{B}}(E, F)$ one has $h \otimes 1_A \in \text{Hom}_{\mathcal{B}}(E \otimes A, F \otimes A)$ for $A \in \text{Ob} \mathcal{A}$. Assume the category $\mathcal{B}$ has this same property. Then we say that $\mathcal{J}$ is natural in the $\mathcal{C}$-variable (or natural in $\mathfrak{g}$) if the following diagram is commutative for all $E, F \in \text{Ob} \mathcal{C}$, $A \in \text{Ob} \mathcal{A}$, and $f \in \text{Hom}_{\mathcal{B}}(E, F)$:

$$
\begin{array}{ccc}
E \otimes \tau A & \xrightarrow{i_{E,A}} & \tau(E \otimes A) \\
\downarrow f \otimes 1_A & & \downarrow \tau(f \otimes 1_A) \\
F \otimes \tau A & \xrightarrow{i_{F,A}} & \tau(F \otimes A)
\end{array}
$$

(1)
We call \( \mathcal{F} \) distributive if the following diagram is commutative for \( E, F \in \text{Ob}\mathcal{F} \) and \( A \in \text{Ob}\mathfrak{A} \):
\[
\begin{array}{ccc}
(E \oplus F) \otimes A & \xrightarrow{i_{E \oplus F, A}} & \tau((E \oplus F) \otimes A) \\
\downarrow & & \downarrow \\
(E \otimes A) \oplus (F \otimes A) & \xrightarrow{i_{E, A} \oplus i_{F, A}} & \tau(E \otimes A) \oplus \tau(F \otimes A)
\end{array}
\]

(2)

The left map (2) expresses the bilinearity of \( \otimes \), and the right map expresses this bilinearity combined with additivity of \( \tau \).

We say that \( \mathcal{F} \) is associative if the following diagram is commutative for all \( E, F \in \text{Ob}\mathcal{F} \) and \( A \in \text{Ob}\mathfrak{A} \):
\[
\begin{array}{ccc}
E \otimes F \otimes A & \xrightarrow{i_{E \otimes F, A}} & E \otimes \tau(F \otimes A) \\
\downarrow & & \downarrow \\
\tau(E \otimes F \otimes A) & \xrightarrow{i_{E, F \otimes A}} & \tau(E \otimes A) \oplus \tau(F \otimes A)
\end{array}
\]

(3)

3.2 Lemma [C]. Suppose \( \mathfrak{A} \) and \( \mathcal{B} \) are \( \mathcal{F} \)-categories and \( \tau : \mathfrak{A} \to \mathcal{B} \) is an intertwining functor. If the family \( \mathcal{F} = \{i_F, A | F \in \text{Ob}\mathcal{F}, A \in \text{Ob}\mathfrak{A}\} \) is natural in the \( \mathcal{F} \)-variable then \( \mathcal{F} \) is distributive.

Suppose \( \tau \) is an intertwining functor with the family of natural equivalences \( \mathcal{F} = \{i_F, F | F \in \text{Ob}\mathcal{F}\} \). We call the pair \( (\tau, \mathcal{F}) \) an \( \mathfrak{A} \)-functor whenever \( \mathcal{F} \) is both distributive and associative. When \( \mathcal{F} \) is understood to be fixed we call \( \tau \) an \( \mathfrak{A} \)-functor. (See [C] and the references listed there for several examples of \( \mathfrak{A} \)-functors in the representation theory of Lie algebras.)

4. Deodhar’s \( \alpha \)-localization functor and new examples of \( \mathfrak{A} \)-functors

4.1. Let \( g \) be a Kac-Moody algebra, and let \( \mathfrak{g} \) be the category of integrable \( g \)-modules. In this section we review Deodhar’s \( \alpha \)-localization functor \( D_\alpha \) for \( \alpha \) a real root of \( g \). Our main result, Theorem 4.18, is a substantial generalization of Deodhar’s results on localization and will provide us with new examples of \( \mathfrak{g} \)-functors. Let us begin by reviewing some noncommutative ring theory (our general reference will by [GW]).

4.2. All rings in this article are assumed to have an identity. If \( X \) is a multiplicative subset then a left ring of fractions for \( R \) with respect to \( X \) is a ring homomorphism \( \phi : R \to S \) such that
(a) \( \phi(x) \) is a unit in \( S \) for all \( x \in X \),
(b) each element of \( S \) can be written in the form \( \phi(x)^{-1}\phi(r) \) for some \( x \in X \) and \( r \in R \), and
(c) \( \ker(\phi) = \{ r \in R | xr = 0 \text{ for some } x \in X \} \).

A multiplicative set \( X \) in a ring \( R \) that satisfies the following two conditions is called a left denominator set:
(Da) \( Xr \cap Rx \neq \emptyset \) for all \( r \in R \) and \( x \in X \);
(Db) if \( r \in R \) and \( x \in X \) are such that \( xr = 0 \) then there exists \( x' \in X \) such that \( rx' = 0 \).

A well-known result due to Goldie is
4.3. **Theorem** [GW, Proposition 9.7]. Let $X$ be a multiplicative set in a ring $R$. Then there exists a left ring of fractions for $R$ with respect to $X$ if and only if $X$ is a left denominator set.

Moreover, if a left ring of fractions exists then it is unique up to isomorphism (see [GW, Corollary 9.5]). If $X$ is a left denominator set then we let $X^{-1}R$ denote its unique ring of fractions. This (as is well known) can be constructed as follows: Define on $X \times R$ an equivalence relation $\sim$ where $(x, r) \sim (x', r')$ if there exists $s \in R$ and $y \in X$ such that $yr = sr'$ and $yx = sx'$. The set of equivalence classes of $X \times R$ has an obvious ring structure which gives us $X^{-1}R$. We let $x \backslash r$ denote the equivalence class of $(x, r)$.

4.4. Let $X$ be a left denominator set, and let $A$ be a left $R$-module. A module of fractions for $A$ with respect to $X$ is an $R$-module homomorphism $f: A \to B$ where $B$ is a left $X^{-1}R$-module such that

(a) every element of $B$ can be written in the form $x^{-1}f(a)$ for some $x \in X$ and $a \in A$, and

(b) $\ker f = \{a \in A | ax = 0 \text{ for some } x \in X\}$.

Another basic fact is

**Theorem** [GW, Corollary 9.11 and Theorem 9.13]. If $X$ is a left denominator set in a ring $R$ then there exists a unique (up to isomorphism) module of fractions for any left $R$-module $A$ with respect to $X$.

4.5. When $X$ is a left denominator set and $A$ is a left $R$-module then we let $X^{-1}A$ denote its unique module of fractions with respect to $X$. In addition there is the following useful construction of $X^{-1}A$.

4.6. **Theorem** [GW, Proposition 9.14]. For $X$ a left denominator set in $R$ and $A$ a left $R$-module we have

$$X^{-1}R \otimes_R A \cong X^{-1}A$$

where the map is given by $s \otimes a \mapsto sa$ for $a \in A$ and $s \in X^{-1}R$.

One has the following universal mapping property of localizations.

**Proposition** [GW, Proposition 9.10]. Let $X$ be a left denominator set in a ring $R$, let $A$ be a left $R$-module, and suppose $C$ is a left $X^{-1}R$-module with $g: A \to C$ an $R$-module homomorphism. If $f: A \to X^{-1}A$ is the module of fractions for $A$ with respect to $X$, then there exists a unique $X^{-1}R$-module homomorphism $h: X^{-1}A \to C$ such that $g = h \circ f$.

4.7. Next we define Deodhar's $\alpha$-localization functor and record the well-known fact that it is just “localization” with respect to the multiplicative set $S_\alpha = \{y_\alpha^n | n \in \mathbb{N}\}$ where $y_\alpha \in g_{-\alpha}$ is nonzero and $\alpha$ is a positive root.

Let $\mathcal{A}_\alpha$ denote the category of $\mathfrak{h}$-semisimple $U(n_-)$-torsionfree $g$-modules. If $A \in \text{Ob} \mathcal{A}_\alpha$, let $A' = \{y_\alpha^{-n} | n \in \mathbb{N}\} \times A$, and define an equivalence relation on $A'$ by $(y_\alpha^{-n}, a) \sim (y_\alpha^{-m}, a')$ if and only if $y_\alpha^n a' = y_\alpha^m a$. Set $D_\alpha(A) = A'/\sim$. $D_\alpha(A)$ is given a $g$-module structure as follows.

4.8. **Lemma** [D, Lemma 2.1]. If $z \in g$ and $0 \leq r \in \mathbb{Z}$ then there exists $0 \leq s \in \mathbb{Z}$ such that $y_\alpha^r z = uy_\alpha^s$ for some $u \in U(g)$.

**Remark.** Deodhar's proof is based on a straightforward $\text{sl}(2, \mathbb{C})$ computation, and from the proof it is easy to see that this lemma is also true for $z \in U(g)$.
so that \( S_\alpha \) is a left Ore set, i.e., it satisfies \((D)\). It then follows from a result of Goldie (see [GW, Proposition 9.9]) that \( S_\alpha \) is a left denominator set.

Now if \((y_\alpha^{\sim}, a) \in D_\alpha(A)\) and \( z \in g \) then \( z(y_\alpha^{\sim}, a) \) is defined to be \((y_\alpha^{\sim}, ua)\) where \( u \) is in the lemma above.

4.9. **Proposition** [D, Proposition 2.2]. Under the action of \( g \) on \( D_\alpha \) given above, \( D_\alpha(A) \) becomes a well-defined \( g \)-module.

**Remark.** By definition \( D_\alpha \) is Deodhar's \( \alpha \)-localization functor.

Next for \( A \in \mathcal{A} \), \( A \leftarrow D_\alpha(A) \) as a \( g \)-module so that by the universal mapping property of the module of fractions there exists a unique \( S_\alpha^{-1}U(g) \)-module map \( S_\alpha^{-1}A \rightarrow D_\alpha A \). In fact this map is given by \( y_\alpha^{-n}a \mapsto (y_\alpha^{-n}, a) \). This map is clearly a bijection. Hence \( S_\alpha^{-1}A \cong D_\alpha \).

4.10. We now turn to our generalization of these results of Deodhar. Let \( X_0 \subset g \oplus \mathbb{C} = U(g)_{1} \), and let \( X \subset U(g) \) be a left denominator set generated by \( 1 \) and \( X_0 \) with the following property: for \( x \in X_0 \) and \( E \) any finite-dimensional \( g \)-module

\[(1) \text{ if } \lambda \text{ is an eigenvalue of the action of } x \text{ on } E \text{ then } x + \lambda \cdot 1 \in X.\]

We shall suppose throughout this section that \( X \) satisfies (1).

For convenience we set \( R = U(g) \) and \( S = X^{-1}R \).

4.11. **Lemma.** Let \( x \in X \). Then \( x \) acts by an isomorphism on \( S \otimes E \).

**Proof.** It is sufficient to check the lemma for \( x \in X_0 \). Introduce an \( x \)-stable filtration \( 0 = E_0 \subset \cdots \subset E_d \) on \( E \) with \( x \) acting by \( \lambda_i \cdot 1 \) on \( E_i/E_{i-1} \). Then suppose \( x \) acts by isomorphism on \( S \otimes E_{i-1} \). Thus for \( \{e_k\} \) a basis of \( E \) with \( e_{ij} \in E_i \)

\[x(s \otimes e_{ij}) \equiv xs \otimes e_{ij} + s \otimes \lambda_i e_{ij} \mod S \otimes E_{i-1}\]

\[\equiv (x + \lambda_i)s \otimes e_{ij} \mod S \otimes E_{i-1}.\]

This proves the lemma.

\( S \) is an \( R \)-bimodule. We shall need other bimodules and so introduce the notation \( E \otimes \mathbb{C} \) to denote the \( R \)-bimodule with left action given on \( E \) and trivial right action. Similarly, let \( E^\sigma \) be the right \( R \)-module defined by \( r(x) \cdot e = \sigma(x) \cdot e, \ e \in E, \ x \in R, \) and \( \sigma \) is the involutive antiautomorphism of \( U(g) \) equal to \( -1 \) on \( g \). Then \( \mathbb{C} \otimes E^\sigma \) is an \( R \)-bimodule with trivial left action and right action given on \( E^\sigma \).

4.12. **Lemma.** As \( S \)-bimodules we have an isomorphism \( S \otimes (E \otimes \mathbb{C}) \cong S \otimes (\mathbb{C} \otimes E^\sigma) \).

**Proof.** Consider the map \( \phi: (R \otimes (E \otimes \mathbb{C})) \rightarrow R \otimes (\mathbb{C} \otimes E^\sigma) \) given by \( x \otimes e \otimes 1 \mapsto (1 \otimes 1 \otimes e) \cdot \sigma(x) \). An easy induction argument using the filtration on \( U(g) \) shows that \( \phi \) is surjective even at the filtered left; i.e., \( \phi: R_i \otimes (E \otimes \mathbb{C}) \rightarrow R_i \otimes (\mathbb{C} \otimes E^\sigma) \) is surjective.

To prove injectivity suppose \( 0 = \sum \lambda_i (1 \otimes 1 \otimes e_i) \cdot \sigma(x_i) \) where the \( e_i \) are as in the proof of Lemma 4.11 and let \( d \) be the maximal integer such that some \( x_j \in R_d \) but \( x_j \notin R_{d-1} \). Then \( 0 \equiv \sum \lambda_i \sigma(x_i) \otimes 1 \otimes e_i \mod R_{d-1} \otimes (\mathbb{C} \otimes E^\sigma) \). However, the \( e_i \) are linearly independent, so each \( \sigma(x_i) \in R_{d-1} \). We clearly have a contradiction. This proves injectivity; thus \( \phi \) is an isomorphism.
Now we extend \( \phi \) to a map \( \phi : S \otimes (E \otimes C) \to S \otimes (C \otimes E^\sigma) \). We must define \( \phi(x \otimes r \otimes e \otimes 1) \). Recall that \( x \) acts on the right of \( S \otimes (C \otimes E^\sigma) \) by an isomorphism \( i(x) \). Set

\[
\phi(x \otimes r \otimes e \otimes 1) = (1 \otimes 1 \otimes e)i(x)^{-1}a(r) = (1 \otimes 1 \otimes e)a(x \otimes r)
\]

where \( a(-) \) denotes the right action of \( R \) on \( S \otimes (C \otimes E^\sigma) \). To see that \( \phi \) is well defined suppose \( x \otimes r = x' \otimes r' \), i.e., there exists \( y \in X \) and \( s \in R \) such that \( yr = sr' \) and \( yx = sx' \). Then we have \( a(yr) = a(sr') \) and \( a(yx) = a(sx') \); thus, \( \phi \) is well defined.

Next we check that \( \phi \) is an \( R \)-bimodule isomorphism. Suppose \( x \in \mathfrak{g} \), and let \( l(x) \) \( (r(x)) \) denote the left (resp. right) action. Then for \( y \in X \), \( z \in R \), \( e \in E \) we have

\[
l(x)\phi(y \otimes z \otimes e \otimes 1) = l(x)(1 \otimes 1 \otimes e)r(y)^{-1}r(z) = (x \otimes 1 \otimes e)r(y)^{-1}r(z)
\]

\[
= (x \otimes 1 \otimes e)r(y)^{-1}r(z) + (1 \otimes 1 \otimes \sigma(x) \cdot e)r(y)^{-1}r(z)
\]

\[
+ (1 \otimes 1 \otimes x \cdot e)r(y)^{-1}r(z)
\]

\[
= (1 \otimes 1 \otimes e)r(x)r(y)^{-1}r(z) + (1 \otimes 1 \otimes x \cdot e)r(y)^{-1}r(z)
\]

\[
= \phi(x(y \otimes z) \otimes e \otimes 1) + \phi(y \otimes z \otimes x \cdot e \otimes 1).
\]

Now for the right action we have

\[
r(x)\phi(y \otimes z \otimes e \otimes 1) = (1 \otimes 1 \otimes e)r(y)^{-1}r(z)r(x)
\]

\[
= \phi(y \otimes zx \otimes e \otimes 1) \phi(y \otimes z \otimes e \otimes 1)r(x).
\]

Finally we check that \( \phi \) is an \( S \)-bimodule map. Suppose \( x \in X \); then from above \( l(x) \circ \phi = \phi \circ l(x) \). By Lemma 4.11 we can invert these left actions. Multiplying out we have \( \phi \circ l(x)^{-1} = l(x)^{-1} \circ \phi \); so \( \phi \) intertwines with the left action of \( S \). Similarly for the right action. This proves the lemma.

**4.13. Lemma.** Let \( F \) be any left \( R \)-module. As left \( S \)-modules we have the isomorphism

\[
S \otimes_R (E \otimes F) \cong S \otimes (C \otimes E^\sigma) \otimes_R F.
\]

**Proof.** The left tensor product above is the quotient of \( S \otimes E \otimes F \) by the relations determined by \( s \cdot x \otimes e \otimes f = s \otimes x \cdot (e \otimes f) \) where \( s \in S \), \( e \in E \), \( f \in F \), and \( x \in \mathfrak{g} \). However, this is precisely the same set of relations as \( (s \otimes e')x \otimes f = s \otimes e' \otimes x \cdot f \) for \( e' \in E^\sigma \). This proves the lemma.

In the next lemma we introduce another multiplicative subset of \( U(\mathfrak{g}) \). We will see later (4.18) that this gives us another example of an \( \mathfrak{h} \)-functor. First we define an \( \mathfrak{h} \)-graded multiplicative subset of \( U(\mathfrak{g}) \) to be a multiplicative subset \( Y \) of \( U(\mathfrak{g}) \) such that \( Y = \bigcup_{\eta \in \mathfrak{h}^*} Y_\eta \) and \( Y_\eta \cdot Y_\beta \subset Y_{\eta + \beta} \) where \( Y_\eta := \{ y \in Y \mid [h, y] = \eta(h)y \ \text{for all} \ h \in \mathfrak{h} \} \).

**4.14. Lemma.** Let \( Y \) be an \( \mathfrak{h} \)-graded multiplicative subset of \( U(n_-) \), \( W \) an \( \mathfrak{h} \)-semisimple \( \mathfrak{g} \)-module, and \( E \) a finite-dimensional \( \mathfrak{g} \)-module. Then \( E \otimes W \) is an \( \mathfrak{h} \)-semisimple \( \mathfrak{g} \)-module.

**Proof.** The fact that \( E \otimes W \) is \( \mathfrak{h} \)-semisimple is obvious. Thus we need only show that \( x(E \otimes W) = E \otimes W \) for all \( x \in Y \). We may assume \( x \) is not an element in \( Y_0 = \{ y \in Y \mid h \cdot y = 0 \ \text{for all} \ h \in \mathfrak{h} \} \). The result will follow if we can show that, for arbitrary \( e \in E_\gamma \) and \( w \in W_\lambda \), \( e \otimes w \) is in \( x(E \otimes W) \). For \( \lambda \in \mathfrak{h}^* \)
let \( E_\lambda \) denote the \( \lambda \)th weight space of \( E \). Since \( E \) is finite dimensional, the set \( \Lambda(E) = \{ \lambda \in h^*|E_\lambda \neq 0 \} \) is finite; hence there exists a minimal \( \mu \in \Lambda(E) \). If \( \gamma \in \Lambda(E) \) is minimal then \( xe = 0 \) for \( e \in E_\gamma \). Now as \( W \) is \( Y \)-divisible there exists for any \( w \in W \) a \( w' \in W \) such that \( xw' = w \) and \( x(e \otimes w') = e \otimes w \).

Thus we may assume as an induction hypothesis that \( \gamma \) is not minimal, and the previous statement is true for all \( e \in E_\mu, w \in W \) with \( \mu < \gamma \). Now if \( e' \in E_\gamma \) and \( w \in W \) then

\[
x(e' \otimes w') = e' \otimes w + \sum_i e'_i \otimes w'_i
\]

where \( e'_i \in E_\mu_i \) with \( \mu_i < \gamma \) and \( xw' = w \). By induction \( e'_i \otimes w'_i \in x(E \otimes W) \); thus \( e' \otimes w \in x(E \otimes W) \). This completes the proof of the lemma.

4.15. Lemma. Let \( Y \) be an \( h \)-graded multiplicative subset of \( U(n_-) \), \( W \) an \( h \)-semisimple \( Y \)-torsionfree \( g \)-module, and \( E \) a finite-dimensional \( g \)-module. Then \( E \otimes W \) is an \( h \)-semisimple \( Y \)-torsionfree \( g \)-module.

Proof. Let \( \sum_i e_{\mu_i} \otimes w_{\lambda_i} \neq 0 \) with \( e_{\mu_i} \in E_{\mu_i}, w_{\lambda_i} \in W_{\lambda_i} \) linearly independent and \( x(\sum_i e_{\mu_i} \otimes w_{\lambda_i}) = 0 \) for some \( x \in Y_{\eta} \) where \( \eta \neq 0 \). Lexicographically order the set \( \Lambda(E) \times \Lambda(W) \) (notation as in the previous lemma), and choose \((\mu_j, \lambda_j)\) maximal with respect to this ordering and such that \( e_{\mu_j} \otimes w_{\lambda_j} \neq 0 \). Then \( 0 = \sum_i e_{\mu_i} \otimes x w_{\lambda_i} + \sum_i e'_{\mu_i} \otimes w'_{\lambda_i} \) where \( \mu'_k < \mu_j \) for all \( k \). But then \( e_{\mu_j} \otimes x w_{\lambda_j} = 0 \); thus \( x w_{\lambda_j} = 0 \). Since \( W \) is \( Y \)-torsionfree, \( x = 0 \). Hence \( E \otimes W \) is \( Y \)-torsionfree.

4.16. Lemma. If \( E \) and \( W \) are \( g \)-modules with \( E \) finite dimensional then

\[
\text{Hom}(E, W) \cong E^* \otimes W.
\]

4.17. Theorem (Mackey Isomorphism Theorem). Suppose \( V \) and \( E \) are \( g \)-modules with \( E \) finite dimensional and \( V \) \( h \)-semisimple. Let \( Y \subseteq U(n_-) \) be an \( h \)-graded left denominator set. Then we have a natural isomorphism

\[
Y^{-1}(V \otimes E) \cong (Y^{-1}V) \otimes E
\]

of \( g \)-modules.

Proof. For the proof of this theorem set \( R = U(n_-) \), and let \( W \) be an \( h \)-semisimple \( Y^{-1}R \)-module. Now consider the following sequence of isomorphisms:

\[
\text{Hom}_{Y^{-1}R}(Y^{-1}(E \otimes V), W) \\
\cong \text{Hom}_{R}(E \otimes V, W) \cong \text{Hom}_{R}(V, \text{Hom}(E, W)) \\
\cong \text{Hom}_{Y^{-1}R}(Y^{-1}V, \text{Hom}(E, W)) \cong \text{Hom}_{Y^{-1}R}(E \otimes Y^{-1}V, W).
\]

The first isomorphism is just given by Proposition 4.6. The second and fourth isomorphisms are derived from the adjoint associativity property of \( \text{Hom} \). Now an \( R \)-module is a \( Y^{-1}R \)-module if and only if it is \( Y \)-divisible and \( Y \)-torsionfree (see [GW, Proposition 9.12]). Thus Lemmas 4.14 through 4.16 imply that \( \text{Hom}(E, W) \) is a \( Y^{-1} \)-module so that we can apply Proposition 4.6 to obtain the third isomorphism. We now proceed as in [Kn, Proposition 5.14] and let \( W = Y^{-1}(E \otimes V) \). The identity in \( \text{Hom}_{Y^{-1}R}(Y^{-1}(E \otimes V), Y^{-1}(E \otimes V)) \) induces a map in \( \text{Hom}_{Y^{-1}R}(E \otimes Y^{-1}V, Y^{-1}(E \otimes Y^{-1}V)) \) which we will denote by \( \varphi \). Similarly the identity morphism in \( \text{Hom}_{Y^{-1}R}(E \otimes Y^{-1}V, E \otimes Y^{-1}V) \) induces a map \( \psi \in \text{Hom}_{Y^{-1}R}(Y^{-1}(E \otimes V), E \otimes Y^{-1}V) \). One can check as in
[Kn, Proposition 5.14] that \( \psi \circ \varphi = 1 \) and \( \varphi \circ \psi = 1 \). This completes the proof of the theorem.

We now consider localization as a functor on the category \( M_\mathfrak{g} \) of all \( \mathfrak{g} \)-modules. Let \( \mathcal{F} \) denote the subcategory of all finite-dimensional \( \mathfrak{g} \)-modules. Set \( \tau_x M = X^{-1}M \) and \( \tau_y M = Y^{-1}M \) where \( X \) satisfies 5.10.1, and \( Y \subseteq U(\mathfrak{n}_-) \) is an \( \mathfrak{h} \)-graded left denominator set. The main result of the article is

4.18. **Theorem.** \( \tau_x \) and \( \tau_y \) are intertwining \( \mathcal{F} \)-functors that are natural in \( \mathcal{F} \).

**Proof.** Theorem 4.17 proves the theorem for \( Y \). For \( X \) we let \( E \in \text{Ob} \mathcal{F} \), \( M \in M_\mathfrak{g} \). Then

\[
\tau_x(E \otimes M) = S \otimes_R (E \otimes M) \cong S \otimes (C \otimes E^\sigma) \otimes_R M \quad \text{(by Lemma 4.13)}
\]

\[
\cong S \otimes (E \otimes C) \otimes_R M \cong E \otimes \tau_x M \quad \text{(by } \phi^{-1} \text{ and Lemma 4.12)}.
\]

The first isomorphism is induced by the identity map, so we shall identify these spaces. For the second isomorphism we have \( \phi^{-1} \) where \( \phi \) is given by \( \phi = \phi_E \) where \( \phi_E(x \setminus r \circ e \otimes 1) = (1 \otimes 1 \otimes e)a(x)^{-1}a(r) \).

Now suppose \( \gamma \in \text{Hom}_\mathfrak{g}(E, F) \); \( E, F \in \text{Ob} \mathcal{F} \). Since \( 1 \otimes \gamma \) intertwines the (right) action of \( \mathfrak{g} \) from \( S \otimes E^\sigma \) to \( S \otimes F^\sigma \), \( 1 \otimes \gamma \) intertwines that of \( a(x)^{-1} \) as well. This implies that the family of equivalences \( \{ \phi_E|E \in \text{Ob} \mathcal{F} \} \) is natural in \( \mathcal{F} \). This proves the theorem.

**Remark.** For the case \( X = S_\alpha \) the result above is due to Deodhar (see [D, Theorem 3.1]) where again the proof is based on an \( \mathfrak{sl}(2, \mathbb{C}) \) calculation.

4.19. **Corollary.** \( \tau_x \) and \( \tau_y \) are \( \mathcal{F} \)-functors.

**Proof.** By Theorem 4.18 and Lemma 3.2 we need only show that \( \tau = \tau_x \) and \( \tau = \tau_y \) are associative. First consider \( \tau = \tau_x \). Suppose \( E, F \in \text{Ob} \mathcal{F} \), \( M \in \text{Ob} M_\mathfrak{g} \). Since the maps involved are all left \( \mathfrak{S} \)-module maps, we need only verify the identity on a set of \( \mathfrak{S} \)-generators for the space. Clearly \( C \otimes C \otimes E^\sigma \otimes C \otimes M \) is a set of generators; so \( C \otimes E \otimes F \otimes C \otimes M \) is a set of \( \mathfrak{S} \)-generators for \( S \otimes E \otimes F \otimes C \otimes M \). But \( \phi_{E \otimes F} \) and also \( \phi_E \circ (1 \otimes \phi_F) \) essentially equal the identity map on these generators;

\[
\phi_{E \otimes F}(1 \otimes e \otimes f \otimes m) = 1 \otimes 1 \otimes e \otimes f \otimes m
\]

\[
= \phi_E \circ (1 \otimes \phi_F)(1 \otimes e \otimes f \otimes 1 \otimes m).
\]

If \( \tau = \tau_y \), one uses the identification \( Y^{-1}R \otimes_R A \cong Y^{-1}A \) of Theorem 4.6 and observes that the isomorphism of 4.17 is given by \( 1 \otimes v \otimes e \mapsto 1 \otimes v \otimes e \) on a set of \( \mathfrak{S} \)-generators of \( Y^{-1}(V \otimes E) \). Now one checks as before that the appropriate identity is satisfied on the set \( C \otimes E \otimes F \otimes A \) of \( \mathfrak{S} \)-generators of \( \tau_y(E \otimes F \otimes A) \).

4.20. **Examples.** Here we describe some multiplicative sets \( X \) other than Deodhar's \( S_\alpha \), for which the hypothesis of 4.18 is satisfied. Set \( \mathbb{Z}_{\geq 0} = \{ k \in \mathbb{Z}|k \geq 0 \} \).

(a) If \( A \) is a \( \mathbb{C} \)-algebra and \( x \in A \) then let \( d_x: A \to A \) denote the \( \mathbb{C} \)-algebra map given by \( d_x(y) = xy - yx \) for all \( y \in A \). For this first example we need Theorem [BR, Satz 2.2]. Let \( A \) be a prime Noetherian \( \mathbb{C} \)-algebra. Suppose \( x \in A \) is an element such that \( d_x \) is locally nilpotent on \( A \). If \( x \) is not
nilpotent, then $x$ is a nonzero divisor in $A$ and the localization $S^{-1}A$ exists where $S = \{x^n | n \in \mathbb{Z}_{\geq 0}\}$.

First let $g = \mathfrak{sl}(3, \mathbb{C})$, $\mathfrak{h}$ a Cartan subalgebra of $g$, $R \subset \mathfrak{h}^*$ the set of nonzero roots, $R^+ \subset R$ a set of positive roots, and $B = \{\alpha, \beta\} \subset R^+$ a basis for $R$. Fix a Chevalley basis $\{x_\gamma, h_\alpha, h_\beta | \gamma \in R\}$ of $g$ where $x_\gamma$ denotes the element in this basis with weight $\gamma \in R$. Let $n_{\pm} = \sum_{\gamma \in R_{\pm}} \mathbb{C}x_\gamma$. For $\gamma \in R$, set $x_{-\gamma} = x_{\gamma}$.

Fix $\epsilon \in \mathbb{C} \setminus \{0, 1\}$, and define $S_\epsilon = \{(y_\alpha y_\beta - \epsilon[y_\alpha, y_\beta])^n | n \in \mathbb{Z}_{\geq 0}\}$. By the Theorem above $S_\epsilon$ is a multiplicative left denominator subset of $U(g)$. It is also clear that $S_\epsilon$ is $\mathfrak{h}$-graded. Consequently $S_\epsilon$ satisfies the hypotheses of 4.18. Our goal is to show that localization with respect to $S_\epsilon$ is not the same as localization with respect to the denominator set $S_\epsilon^\mu := \{\mu y_\gamma | n \in \mathbb{Z}_{\geq 0}\}$ for any $\gamma \in R$ and $\mu \in \mathbb{C} \setminus \{0\}$.

We first consider the case that $\gamma = \alpha$. Suppose now that localization with respect to $S_\epsilon^\alpha$ is the same as localization with respect to $S_\epsilon$. Then $y_\alpha$ is invertible in $S_\epsilon^{-1}U(g)$; thus, $(s \setminus m)y_\alpha = 1$ for some $s \in S_\epsilon$ and $m \in U(g)$. Consequently there exists $a \in U(g)$ and $b \in S_\epsilon$ with $as = b \in S_\epsilon$ and $amy_\alpha = b$. This implies that $m y_\alpha = s = (y_\alpha y_\beta - \epsilon[y_\alpha, y_\beta])^n$ for some $n \geq 0$ ($a \neq 0$ since otherwise $0 = as = b \in S_\epsilon$).

Define an ordering of the Chevalley basis by

$$x_\alpha < x_\beta < x_{\alpha+\beta} < h_\alpha < h_\beta < y_{\alpha+\beta} < y_\beta < y_\alpha.$$  

Then the Poincaré-Birkhoff-Witt Theorem tells us that $U(g)$ has a basis of monomials $x^\mu y^\nu$ where

$$\bar{x} = x_\alpha^{n_1}x_\beta^{n_2}x_{\alpha+\beta}^{n_3}, \quad \bar{h} = h_\alpha^{n_4}h_\beta^{n_5}, \quad \bar{y} = y_\alpha^{n_6}y_\beta^{n_7}y_{\alpha+\beta}^{n_8},$$

and $n_i \in \mathbb{Z}_{\geq 0}$. Consequently $m = \sum u_\bar{n}x^\mu y^\nu$ where $u_\bar{n} \in \mathbb{C}$ and $\bar{n} = (n_1, \ldots, n_8) \in \mathbb{Z}_{\geq 0}^8$.

Thus

$$m y_\alpha = \sum u_\bar{n} x^\mu y^\nu y_\alpha = (y_\alpha y_\beta - \epsilon[y_\alpha, y_\beta])^n \in U(n_\cdot)_{n_\cdot}.$$  

By the Poincaré-Birkhoff-Witt Theorem we have that $m \in U(n_\cdot)_{n_\cdot}$. A straightforward calculation shows that

$$m y_\alpha = \sum u_\bar{n} x^\mu y^\nu y_\alpha = (1 - \epsilon)^n y_{\alpha+\beta} + p(y_\alpha, y_\beta, y_{\alpha+\beta})y_\alpha$$

for some polynomial $p$. Thus

$$(m - p(y_\alpha, y_\beta, y_{\alpha+\beta}))y_\alpha = (1 - \epsilon)^n y_{\alpha+\beta}.$$  

If $\epsilon \neq 1$, this is impossible by the Poincaré-Birkhoff-Witt Theorem. This proves that localization with respect to $S_\epsilon^\mu$ is not the same as localization with respect to $S_\epsilon$ if $\epsilon \neq 1$. A very similar argument (but with a different ordering on the basis) shows that localization with respect to $S_\epsilon^\mu$ where $\gamma \in \{\pm \alpha, \pm \beta, \pm(\alpha+\beta)\}$ is not the same as localization with respect to $S_\epsilon$ if $\epsilon \neq 0, 1$.

(b) Let $g = \mathfrak{sl}(2, \mathbb{C})$ with $\{x, h, y\}$ a Chevalley basis of $g$ such that $[h, x] = 2x$, $[x, y] = h$, and $[h, y] = -2y$. Let $X_0 = \{h - n_1 | n \in \mathbb{Z}\}$ and $X = \{(h - n_1)^k | n, k \in \mathbb{Z}, k \geq 0\}$ be subsets of $U(g)$. This is Example 1.8 in [BR] of an Ore subset of $U(g)$. Now it is straightforward to check that $X$ satisfies condition 4.10(1). Thus we only need to see that localization with respect to $X$ is not the same as localization with respect to $S_\alpha = \{x^n | n \in \mathbb{Z}_{\geq 0}\}$ or

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$S_{-\alpha} = \{y^n | n \in \mathbb{Z}_{\geq 0}\}$. We will prove the case $S_{\alpha} = \{x^n | n \in \mathbb{Z}_{\geq 0}\}$ and leave the other case to the reader. As in the previous example we assume the contrapositive so that there exists $m \in U(\text{sl}(2, \mathbb{C}))$, $n_1, \ldots, n_t \in \mathbb{Z}$, and $k_1, \ldots, k_t \in \mathbb{Z}_{>0}$ such that

$$my = (h - n_1)^{k_1} \cdots (h - n_t)^{k_t}.$$ 

Using the Poincaré-Birkhoff-Witt Theorem we see that this is impossible. Thus, localization with respect to $X$ is not the same as localization with $x$ or $y$.

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**References**


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