ON THE FACTORIZATION OF $A_p$ WEIGHTS

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Abstract. It is shown that if $(u, v) \in A_p$ then there exist $(u_1, v_1) \in A_1$ and $(u_2, v_2) \in A_1$ such that $u = u_1^{\delta} v_2^{1-\delta}$, $v = v_1^{\delta} u_2^{1-\delta}$.

1. Introduction

Let $M$ denote the Hardy-Littlewood maximal operator in $\mathbb{R}^n$,

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|,$$

where $Q$ is always a cube in $\mathbb{R}^n$ and $|\cdot|$ denotes Lebesgue measure. Let $w(x)$, $u(x)$, and $v(x)$ be nonnegative measurable functions on $\mathbb{R}^n$, and recall that

$$W \in A_1 \quad \text{if and only if} \quad Mw(x) \leq Cw(x) \text{ a.e. } x \in \mathbb{R}^n;$$

$$w \in A_p \quad \text{if and only if} \quad \left( \int_Q w \right) \left( \int_Q w^{1-p'} \right)^{p-1} \leq C|Q|^p;$$

$$(u, v) \in A_1 \quad \text{if and only if} \quad Mu(x) \leq Cv(x) \text{ a.e. } x \in \mathbb{R}^n;$$

$$(u, v) \in A_p \quad \text{if and only if} \quad \left( \int_Q u \right) \left( \int_Q v^{1-p'} \right)^{p-1} \leq C|Q|^p;$$

$$(u, v) \in S_p \quad \text{if and only if} \quad \left( \int |Mf|^p u \right) \leq C \int |f|^p v.$$

Throughout $C$ denotes a constant and may vary from line to line, $1 < p < \infty$ (unless otherwise noted), and $\frac{1}{p} + \frac{1}{p'} = 1$.

If there exist $C_1 > 0$ and $C_2 > 0$ such that $C_1 u \leq v \leq C_2 u$ a.e. $x \in \mathbb{R}^n$, then $u \sim v$.

In [2] Jones showed that $w \in A_p$ if and only if there exist weights $w_1$, $w_2 \in A_1$ with $w = w_1 w_2^{1-p}$. In this note we consider the factorization of the general $A_p$ weights. Our results are as follows.

Theorem 1. Let $(u, v) \in A_p$ and $0 < \delta < 1$. Then there exist $(u_1, v_1)$ and $(u_2, v_2) \in A_1$ such that

$$u^\delta = u_1^{\delta} v_2^{1-\delta}, \quad v^\delta = v_1^{\delta} u_2^{1-\delta}.$$
Moreover, if \( u, v \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( 0 < u, v < \infty \) a.e. \( x \in \mathbb{R}^n \), then \( u \sim v \) if and only if \( u_1 \sim v_1 \) and \( u_2 \sim v_2 \).

**Theorem 2.** There exist \((u_1, v_1) \in A_1\) and \((u_2, v_2) \in A_1\) such that for any \(0 < \delta < 1\)
\[
(u, v) = ((u_1v_2^{1-p})^{1/\delta}, (v_1u_2^{1-p})^{1/\delta}) \notin A_p.
\]

This shows that the converse of Theorem 1 is false.

**Corollary 1.** If \((u, v) \in A_p\), then there exist \((u_1, v_1)\) and \((u_2, v_2)\) in \(A_1\) such that
\[
u = u_1^{p'}v_2^{-p'}, \quad v = v_1^{p'}v_2^{-p'}.
\]

**Proof.** Using Theorem 1 with \(\delta = 1/p'\), we get the result.

**Corollary 2.** If \(w \in A_p\), then there exist \(w_1, w_2 \in A_1\) such that \(w = w_1w_2^{1-p}\).

**Proof.** We may assume that \(0 < w < \infty\) a.e. so that \(w \in L^1_{\text{loc}}(\mathbb{R}^n)\). Since \(w \in A_p\), there exists \(\tau > 1\) such that \(w^\tau \in A_p\) [1]. Let \(\delta = 1/\tau\). By Theorem 1 there exist \((u_1, v_1), (u_2, v_2) \in A_1\) with \(u_1 \sim v_1, u_2 \sim v_2\), and therefore \(u_1, u_2 \in A_1\), such that \((w^\tau)^{1/\tau} = u_1v_2^{1-p}\), i.e., \(w = u_1v_2^{1-p}\). Let \(u_1 = w_1, v_2 = w_2\); we get \(w = w_1w_2^{1-p}\) with \(w_1, w_2 \in A_1\).

## 2. Proof of the theorems

We need the following lemmas.

**Lemma 1** [3]. If \((u, v) \in A_p\), and \(0 < \delta < 1\), then \((u^\delta, v^\delta) \in S_p\).

**Lemma 2** [3]. Assume that \((u, v) \in S_p\) and \((v^{1-p'}, u^{1-p'}) \in S_{p'}\). Then there are functions \(w_j \geq 0\) such that
\[
u^{1/p}Mw_j \leq C_jw_jv^{1/p}, \quad j = 1, 2,
\]
and
\[
u^{1/p}v^{1/p'} = w_1w_2^{1-p}.
\]

**Lemma 3.** If \((u, v) \in A_p\) (\(1 \leq p < \infty\)) and \(u, v \in L^1_{\text{loc}}(\mathbb{R}^n)\), then \(u(x) \leq Cv(x)\) a.e. \(x \in \mathbb{R}^n\).

**Proof.** For \(p = 1\) it is obvious. For \(1 < p < \infty\), since \((u, v) \in A_p\) if and only if [1]
\[
\left(\frac{1}{|Q|}\int_Q f \right)^p \left(\int_Q u \right) \leq C \int_Q f^pv
\]
for any \(f \geq 0, Q \subset \mathbb{R}^n\), let \(f = 1\). Then \(\int_Q u \leq C \int_Q v\), i.e.,
\[
\frac{1}{|Q|}\int_Q u \leq C \frac{1}{|Q|}\int_Q v.
\]
Let \(|Q| \to 0\). Since \(u, v \in L^1_{\text{loc}}(\mathbb{R}^n)\) we get
\[
u(x) \leq Cv(x) \quad \text{a.e. } x \in \mathbb{R}^n.
\]

**Lemma 4.** If \((u, v) \in S_p\) and \((v^{1-p'}, u^{1-p'}) \in S_{p'}\), then there exist \((u_j, v_j) \in A_1\), \(j = 1, 2\), such that
\[
u = u_1v_2^{1-p}, \quad v = v_1u_2^{1-p}.
\]
Proof. Let $u_j = w_j$ and $v_j = w_j v^{1/p} u^{-1/p}$ by Lemma 2. Then $(u_j, v_j) \in A_1$, $j = 1, 2$.

Since $u^{1/p} v^{1/p'} = w_1 w_2^{1-p}$, we get

$$u = w_1^{p(1-p)} w_2^{1-p}, \quad v = w_1^{p'} w_2^{1-p} u^{1-p'}.$$

Thus,

$$u_1 v_2^{1-p} = w_1 (w_2 v^{1/p} u^{-1/p})^{1-p} = w_1 w_2^{1-p} v^{1-p'} w_1^{p(1-p)} v^{1-p'} u^{1-p'} = w_1 w_2^{1-p} = u$$

and

$$v_1 u_2^{1-p} = (w_1 v^{1/p} u^{-1/p}) w_2^{1-p} = w_1 w_2^{1-p} w_2^{1-p} u^{1-p'} = w_1^{p'} w_2^{1-p} u^{1-p'} = v.$$

Proof of Theorem 1. Since $(u, v) \in A_p$ if and only if $(v^{1-p'}, u^{1-p'}) \in A_{p'}$. Lemma 1 gives $(u^\delta, v^\delta) \in S_p$ and $(v^{\delta(1-p')}, u^{\delta(1-p')}) \in S_{p'}$. From Lemma 4 we get $(u_j, v_j) \in A_1$, $j = 1, 2$, such that

$$u^\delta = u_1 v_2^{1-p}, \quad v^\delta = v_1 u_2^{1-p}.$$

For $u, v \in L_1^{\text{loc}}(R^n)$ and $0 < u, v < \infty$ a.e. if $u_1 \sim v_1$, $u_2 \sim v_2$, then $u^{1-p}_1 \sim u^{1-p}_2$. Hence $u^{\delta} \sim v^\delta$, i.e., $u \sim v$.

Conversely, if $u \sim v$, then there exist $C_1 > 0$, $C_2 > 0$ such that

$$C_1 u^{1-p}_1 \leq u^{1-p}_2 \leq C_2 v^{1-p}_1.$$

From the left inequality, we have

$$(v_2/u_2)^{p-1} \leq C u_1/v_1.$$

Note that $u_j \leq C v_j$, $j = 1, 2$. We get $u_2 \sim v_2$ and also $u_1 \sim v_1$.

Proof of Theorem 2. Let $u_1 = v_1 \equiv 1$, $u_2 = \varphi(x) = \chi_{[0,1]}$, and $v_2 = M \varphi$. Then $(u_j, v_j) \in A_1$, $j = 1, 2$, but $(u_1 v_1^{1-p}, v_1 u_2^{1-p}) = ((M \varphi)^{1-p}, \varphi^{1-p}) \notin S_p$. In fact, let $f = \varphi = \chi_{[0,1]}$. Then we have

$$\int (M f)^{p(1-p)} = \int M \varphi = +\infty,$$

but $\int |f|^{p} \varphi^{1-p} = \int \varphi = 1$.

From Lemma 1, $(u, v) = ((u_1 v_2^{1-p})^{1/\delta}, (v_1 u_2^{1-p})^{1/\delta}) \notin A_p$ for any $0 < \delta < 1$.

References


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