ON THE FACTORIZATION OF $A_p$ WEIGHTS

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ABSTRACT. It is shown that if $(u, v) \in A_p$ then there exist $(u_1, v_1) \in A_1$ and $(u_2, v_2) \in A_1$ such that $u = u_1^\delta v_2^{1-p}$, $v = v_1^\delta u_2^{1-p}$.

1. Introduction

Let $M$ denote the Hardy-Littlewood maximal operator in $\mathbb{R}^n$, 

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|,$$

where $Q$ is always a cube in $\mathbb{R}^n$ and $|\cdot|$ denotes Lebesgue measure. Let $w(x)$, $u(x)$, and $v(x)$ be nonnegative measurable functions on $\mathbb{R}^n$, and recall that

$$W \in A_1 \quad \text{if and only if} \quad Mw(x) \leq Cw(x) \quad \text{a.e.} \quad x \in \mathbb{R}^n;$$

$$w \in A_p \quad \text{if and only if} \quad \left( \int_Q w \right) \left( \int_Q w^{1-p'} \right)^{p-1} \leq C|Q|^p;$$

$$(u, v) \in A_1 \quad \text{if and only if} \quad Mu(x) \leq Cv(x) \quad \text{a.e.} \quad x \in \mathbb{R}^n;$$

$$(u, v) \in A_p \quad \text{if and only if} \quad \left( \int_Q u \right) \left( \int_Q v^{1-p'} \right)^{p-1} \leq C|Q|^p;$$

$$(u, v) \in S_p \quad \text{if and only if} \quad \int |Mf|^p u \leq C \int |f|^p v.$$ 

Throughout $C$ denotes a constant and may vary from line to line, $1 < p < \infty$ (unless otherwise noted), and $\frac{1}{p} + \frac{1}{p'} = 1$.

If there exist $C_1 > 0$ and $C_2 > 0$ such that $C_1 u \leq v \leq C_2 u$ a.e. $x \in \mathbb{R}^n$, then $u \sim v$.

In [2] Jones showed that $w \in A_p$ if and only if there exist weights $w_1$, $w_2 \in A_1$ with $w = w_1 w_2^{1-p}$. In this note we consider the factorization of the general $A_p$ weights. Our results are as follows.

Theorem 1. Let $(u, v) \in A_p$ and $0 < \delta < 1$. Then there exist $(u_1, v_1)$ and $(u_2, v_2) \in A_1$ such that

$$u^\delta = u_1 v_2^{1-p}, \quad v^\delta = v_1 u_2^{1-p}.$$
Moreover, if \( u, v \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( 0 < u, v < \infty \) a.e. \( x \in \mathbb{R}^n \), then \( u \sim v \) if and only if \( u_1 \sim v_1 \) and \( u_2 \sim v_2 \).

**Theorem 2.** There exist \((u_1, v_1) \in A_1\) and \((u_2, v_2) \in A_1\) such that for any \( 0 < \delta < 1 \)

\[
(u, v) = ((u_1v_2^{1-p})^{1/\delta}, (v_1u_2^{1-p})^{1/\delta}) \notin A_p.
\]

This shows that the converse of Theorem 1 is false.

**Corollary 1.** If \((u, v) \in A_p\), then there exist \((u_1, v_1)\) and \((u_2, v_2) \in A_1\) such that

\[
u = u_1^{p'}v_2^{-p}, \quad v = v_1^{p'}v_2^{-p}.
\]

**Proof.** Using Theorem 1 with \( \delta = 1/p' \), we get the result.

**Corollary 2.** If \( w \in A_p \), then there exist \( w_1, w_2 \in A_1 \) such that \( w = w_1w_2^{1-p} \).

**Proof.** We may assume that \( 0 < w < \infty \) a.e. so that \( w \in L^1_{\text{loc}}(\mathbb{R}^n) \). Since \( w \in A_p \), there exists \( \tau > 1 \) such that \( w^\tau \in A_p \) [1]. Let \( \delta = 1/\tau \). By Theorem 1 there exist \((u_1, v_1), (u_2, v_2) \in A_1\) with \( u_1 \sim v_1, u_2 \sim v_2 \), and therefore \( u_1, u_2 \in A_1 \), such that \((w^\tau)^{1/\tau} = u_1v_2^{1-p}\), i.e., \( w = u_1v_2^{1-p} \). Let \( u_1 = w_1, v_2 = w_2 \); we get \( w = w_1w_2^{1-p} \) with \( w_1, w_2 \in A_1 \).

2. **Proof of the theorems**

We need the following lemmas.

**Lemma 1** [3]. If \((u, v) \in A_p\), and \( 0 < \delta < 1 \), then \((u^\delta, v^\delta) \in S_p\).

**Lemma 2** [3]. Assume that \((u, v) \in S_p\) and \((u^{1-p'}, v^{1-p}) \in S_{p'}\). Then there are functions \( w_j \geq 0 \) such that

\[
u^{1/p}Mw_j \leq C_jw_jv^{1/p}, \quad j = 1, 2,
\]

and

\[
u^{1/p}v^{1/p'} = w_1w_2^{1-p}.
\]

**Lemma 3.** If \((u, v) \in A_p \) \((1 \leq p < \infty)\) and \( u, v \in L^1_{\text{loc}}(\mathbb{R}^n) \), then \( u(x) \leq Cv(x) \) a.e. \( x \in \mathbb{R}^n \).

**Proof.** For \( p = 1 \) it is obvious. For \( 1 < p < \infty \), since \((u, v) \in A_p\) if and only if [1]

\[
\left( \frac{1}{|Q|} \int_Q f \right)^p \left( \int_Q u \right) \leq C \int_Q f^pv
\]

for any \( f \geq 0, Q \subset \mathbb{R}^n \), let \( f \equiv 1 \). Then \( \int_Q u \leq C \int_Q v \), i.e.,

\[
\frac{1}{|Q|} \int_Q u \leq C \frac{1}{|Q|} \int_Q v.
\]

Let \( |Q| \to 0 \). Since \( u, v \in L^1_{\text{loc}}(\mathbb{R}^n) \) we get

\[
u(x) \leq Cv(x) \quad \text{a.e. } x \in \mathbb{R}^n.
\]

**Lemma 4.** If \((u, v) \in S_p\) and \((u^{1-p'}, v^{1-p}) \in S_{p'}, then there exist \((u_j, v_j) \in A_1, j = 1, 2\), such that

\[
u = u_1v_2^{1-p}, \quad v = v_1u_2^{1-p}.
\]
Proof. Let \( u_j = w_j \) and \( v_j = w_j v_j^{1/p} u_j^{-1/p} \) by Lemma 2. Then \( (u_j, v_j) \in A_1, j = 1, 2 \).

Since \( u^{1/p}v^{1/p} = w_1 w_2^{-1} \), we get

\[
u = w_1^p w_2^{p(1-p)}v_1^{1-p}, \quad v = w_1^{p'} w_2^{-p}u_1^{1-p'}.
\]

Thus,

\[
u_1 u_2^{-1} = \frac{w_1}{w_2} (w_2^{1/p} u_1^{-1/p}) u_2^{1-p} = w_2^{1/p} u_1^{1-p} = u
\]

and

\[
u_1 v_1^{-1} = (w_1 v_1^{1/p} u_1^{-1/p}) w_2^{1-p} = w_1^{-1/p} (w_2^{1/p} u_1^{-1/p} u_2^{-1-p'}) w_2^{1-p} = w_2^{1/p} u_2^{-1-p} = v.
\]

Proof of Theorem 1. Since \( (u, v) \in A_p \) if and only if \( (v^{1-p'}, u^{1-p'}) \in A_{p'} \). Lemma 1 gives \( (u^\delta, v^\delta) \in S_p \) and \( (v^{(1-p')\delta}, u^{(1-p')\delta}) \in S_{p'} \). From Lemma 4 we get \( (u_j, v_j) \in A_1, j = 1, 2 \), such that

\[
u^\delta = u_1 v_1^{1-p}, \quad v^\delta = v_1 u_2^{1-p}.
\]

For \( u, v \in L^1_{\text{loc}}(R^n) \) and \( 0 < u, v < \infty \) a.e. if \( u_1 \sim v_1, u_2 \sim v_2 \), then

\( u_1 v_2^{1-p} \sim v_1 u_2^{-1-p} \). Hence \( u^\delta \sim v^\delta \), i.e., \( u \sim v \).

Conversely, if \( u \sim v \), then there exist \( C_1 > 0, C_2 > 0 \) such that

\[
C_1 u_1 v_2^{-1-p} \leq u_1 v_2^{1-p} \leq C_2 v_1 u_2^{1-p}.
\]

From the left inequality, we have

\[
(v_2/u_2)^{p-1} \leq C u_1/v_1.
\]

Note that \( u_j \leq C v_j, j = 1, 2 \). We get \( u_2 \sim v_2 \) and also \( u_1 \sim v_1 \).

Proof of Theorem 2. Let \( u_1 = v_1 \equiv 1, u_2 = \varphi(x) = \chi_{[0, 1]}, \) and \( v_2 = M \varphi \). Then \( (u_j, v_j) \in A_1, j = 1, 2 \), but \( (u_1 v_2^{1-p}, v_1 u_2^{1-p}) = ((M \varphi)^{1-p}, \varphi^{1-p}) \notin S_p \). In fact, let \( f = \varphi = \chi_{[0, 1]} \). Then we have

\[
\int (M f)^p(M \varphi)^{1-p} = \int M \varphi = +\infty,
\]

but \( \int |f|^p \varphi^{1-p} = \int \varphi = 1 \).

From Lemma 1, \( (u, v) = ((u_1 v_2^{1-p})^{\delta}, (v_1 u_2^{1-p})^{\delta}) \notin A_p \) for any \( 0 < \delta < 1 \).

References

