RICCI CURVATURE AND HOLOMORPHIC CONVEXITY IN KÄHLER MANIFOLDS

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ABSTRACT. In this paper we show that if a smoothly bounded, relatively compact domain in a complex manifold admits a complete Kähler metric with certain bounds on its Ricci tensor, then the domain must be holomorphically convex. This gives an obstruction for the existence of a complete Kähler-Einstein metric on such domains.

INTRODUCTION

The existence of a complete Kähler-Einstein metric imposes some strong analytic conditions on an open manifold. Mok and Yau obtained a very satisfying result on the relationship between Kähler-Einstein metrics and holomorphic convexity for a bounded Riemann domain Ω [6]. They proved that the following are equivalent:

(1) Ω is a domain of holomorphy (holomorphically convex).
(2) Ω admits a complete Kähler-Einstein metric of negative Ricci curvature.
(3) Ω admits a complete Hermitian metric with Ricci curvature nonpositive and bounded below.

The paper of Mok and Yau completes and extends earlier work of Cheng and Yau [3].

In this paper we first give two examples which demonstrate that the existence of a Kähler-Einstein metric on an open manifold does not in general imply holomorphic convexity. We then state and prove Theorem 1, which says that for a smoothly bounded, relatively compact domain in a complex manifold, holomorphic convexity is necessary for the existence of a complete Kähler metric with certain bounds on its Ricci tensor.

OPEN KÄHLER-EINSTEIN MANIFOLDS
WHICH ARE NOT HOLOMORPHICALLY CONVEX

First, let $X = \mathbb{C}^2/\Lambda$, where Λ is the group generated by $(1, 0)$, $(0, 1)$, and (a, b) over $\mathbb{Z}$. Then X inherits a complete Ricci-flat metric from $\mathbb{C}^2$.

However, if a and b are complex numbers such that 1, a, and b are linearly independent over $\mathbb{Z}$, then X is not holomorphically convex (see [10]).

One may also construct an example with negative Ricci curvature: let $X = \mathbb{B}^n/\Gamma$, where $\mathbb{B}^n$ is the unit ball with Poincaré metric in $\mathbb{C}^n$ and Γ is an arithmetically defined discontinuous group of isometries of $\mathbb{B}^n$. Then X inherits a complete Kähler-Einstein metric of negative Ricci curvature from $\mathbb{B}^n$. The Satake-Baily-Borel compactification of X is a normal analytic space $\overline{X}$ obtained from X by adjoining a finite number of points (see [1]). If $n \geq 2$, then the normality of $\overline{X}$ implies that holomorphic functions on X extend to $\overline{X}$ and therefore are constant, so X is not holomorphically convex. Because X is not a bounded Riemann domain, this does not contradict the result of Mok and Yau.

This second example differs markedly from the domains considered by Cheng, Mok, and Yau in that the Kähler-Einstein metric on X has finite volume. The complete Kähler-Einstein metric on a bounded Riemann domain of holomorphy has infinite volume; in fact, the coefficient of the volume form grows at least as fast as $d^{-2}(-\log d)^{-2}$ [3, 6].

We note that Mok and Zhong have generalized the Satake-Baily-Borel result; they showed that, under a mild topological condition, a complete Kähler manifold with finite volume, negative Ricci curvature, and bounded sectional curvatures is biholomorphic to a quasi-projective manifold [7]. See [5, 8, 9, 13, 16].

NOTATION

Henceforth Ω will denote a relatively compact domain in a complex manifold M. We denote the volume form associated to a Hermitian metric g by V_g. For a volume form V which equals $(\frac{i}{2})^n v(z) dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$ in local coordinates, the Ricci form $\mathrm{Ric}(V)$ is the globally defined $(1, 1)$-form $\frac{i}{2}\partial\overline{\partial} \log v$.

PRELIMINARY OBSERVATIONS

The existence of a complete Kähler metric on Ω with bounds on its Ricci form immediately imposes some analytic conditions on Ω. A theorem of Ohsawa says that a smoothly (C^1) bounded domain which carries a complete Kähler metric must have pseudoconvex boundary [12]. It need not, however, be holomorphically convex [4]. If Ω admits a Hermitian metric g and a volume form V such that $\mathrm{Ric}(V_g) < \mathrm{Ric}(V)$ (resp. $\mathrm{Ric}(V_g) > \mathrm{Ric}(V)$), then Ω carries a strictly plurisubharmonic function $\log(V_g/V)$ (resp. $\log(V/V_g)$). In particular, Ω may not contain any compact analytic set. For example, if V is the volume form of a Ricci flat metric on M and Ω admits a metric of negative Ricci curvature, then Ω must admit a strictly plurisubharmonic function.

STATEMENT OF THE THEOREM

Theorem 1. *Let M be a complex manifold and $\Omega \Subset M$ a relatively compact domain with C^2 boundary. Suppose that Ω admits a complete Kähler metric g whose Ricci curvature is bounded below and such that*

    (∗) *there is a neighborhood T of $\partial\Omega$ in M and a volume form V_M on T*

such that, for some positive definite $(1, 1)$-*form* ω *defined on* T,

$$\mathrm{Ric}(V_M) - \mathrm{Ric}(V_g) > \omega$$

on $T \cap \Omega$.

Then Ω *is holomorphically convex. If* V_M *is defined on a neighborhood of* $\bar{\Omega}$ *and* (*) *holds on all of* Ω, *then* Ω *is Stein.*

This theorem very nearly proves a conjecture of Narasimhan. He conjectured that if D is a domain with smooth boundary in a projective manifold M, if $c_1(M) \geq 0$, and if D supports a complete Kähler metric with Ricci curvature bounded between negative constants, then D is Stein. The above theorem proves this conjecture if one adds to Narasimhan's conjecture the hypothesis that locally near ∂D the complete Kähler metric dominates a multiple of a Hermitian metric on M.

TECHNICAL LEMMA

We first prove a technical lemma. Let X be a complete Riemannian manifold. Fix $p \in X$, and let $S_k = \{x \in X : d(x, p) \geq k\}$, where $d(x, p)$ denotes the distance from x to p. Let f be a continuous real-valued function on S_{k_0}. Let Γ_k denote the graph of f over S_k, $k \geq k_0$. Γ_k is a closed submanifold (with boundary) of $\mathbb{R} \times X$. Let $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$ be strictly increasing, and let $p_{\sigma(k)}$ denote the point $(\sigma(k), p)$ in $\mathbb{R} \times X$. Because Γ_k is closed and $\mathbb{R} \times X$ is complete with respect to the product metric, there is a geodesic segment g_k in $\mathbb{R} \times X$ from $p_{\sigma(k)}$ to $(f(q_k), q_k) \in \Gamma_k$ whose length is the distance from $p_{\sigma(k)}$ to Γ_k.

Lemma 1. *One of the following is true*:

    (a) *There is some* σ *such that* q_k *may be chosen in the interior* $\overset{\circ}{S}_k$ *of* S_k *for all* k *large enough.*

    (b) *There is a point* $q \in X$ *such that if* $d(x, p) > d(q, p)$, *then* $f(x) \leq f(q)$.

Proof of Lemma 1. Suppose (a) is false. Then there is some k such that q_k must be on ∂S_k for all $\sigma(k)$. Since ∂S_k is compact, f achieves its maximum on ∂S_k at some point which we denote by q. Now

$$d\left(p_{\sigma(k)}, (f(q_k), q_k)\right) = \sqrt{(\sigma(k) - f(q_k))^2 + d(p, q_k)^2}$$
$$= \sqrt{(\sigma(k) - f(q_k))^2 + k^2},$$

which is minimized when $(\sigma(k) - f(q_k))^2$ is minimized. Therefore, one may choose $q_k = q$ for all σ such that $\sigma(k) \geq f(q)$. Suppose there were a point $x \in \overset{\circ}{S}_k$ such that $d(x, p) > k$ and $f(x) > f(q)$. Since $d(p_{\sigma(k)}, (f(x), x)) = \sqrt{(\sigma(k) - f(x))^2 + d(x, p)^2}$ and $d(p_{\sigma(k)}, (f(q), q)) = \sqrt{(\sigma(k) - f(q))^2 + k^2}$, the inequality

$$d\left(p_{\sigma(k)}, (f(x), x)\right) < d\left(p_{\sigma(k)}, (f(q), q)\right)$$

is equivalent to the inequality

(1) $$f(x)^2 - f(q)^2 + d(x, p)^2 - k^2 < 2\sigma(k)(f(x) - f(q)).$$

Inequality (1) is true for $\sigma(k)$ large enough; this contradicts the assumption that q_k must be on ∂S_k for all σ. Therefore, $f(x) \leq f(q)$ for all x such that $d(x, p) > k$, which proves (b). $\square$

Proof of the Theorem

For $z \in \Omega$, let $d(z)$ denote the distance from z to $\partial\Omega$, measured in a fixed metric on M, shrinking T if necessary so that d is C^2 on $T \cap \Omega$. Because Ω carries a complete Kähler metric, it must have pseudoconvex boundary. Thus $i\partial\bar\partial(-\log d(z))$ is bounded below near $\partial\Omega$ (see, e.g., [14]). For k large enough,

$$\tilde\phi(z) = k \log \frac{V_g}{V_M} - \log d(z)$$

is strictly plurisubharmonic in $T \cap \Omega$. We extend $\tilde\phi$ to a continuous plurisubharmonic function ϕ which equals $\tilde\phi$ on $T' \cap \Omega$, T' a relatively compact neighborhood of $\partial\Omega$ in T. To show that Ω is holomorphically convex, we need only show that ϕ is an exhaustion function, that is, that $\{z \in \Omega : \phi(z) \leq a\}$ is compact for all real a (see [11]). Clearly ϕ is an exhaustion function if V_g/V_M is bounded away from zero.

To show that this is the case, we cover $\partial\Omega$ by coordinate balls B_α with Poincaré metric g_α and associated volume form V_α. We compare V_g to V_α on $B'_\alpha \cap \Omega$, where $B'_\alpha \Subset B_\alpha$ and $\partial\Omega \subset \bigcup_\alpha B'_\alpha$. In [6], Mok and Yau prove the following Schwarz lemma for volume forms:

Lemma 2. *Let X be a complete Kähler manifold with Ricci curvature bounded below and let Y be a complex manifold of the same dimension with a volume form V_Y such that $\mathrm{Ric}(V_Y)$ is negative definite and $(-\mathrm{Ric}(V_Y))^n \geq cV_Y$, $c > 0$. Suppose $f : X \to Y$ is a holomorphic map and the Jacobian is nonvanishing at one point. Then $(f^*V_Y)/V_X$ is bounded above on X.*

We would like to apply this theorem to our situation, with $X = Y = B'_\alpha \cap \Omega$, $V_X = V_g$, $V_Y = V_\alpha$, and $f =$ identity. However, we may not apply the lemma directly because the metric g is not complete on $B'_\alpha \cap \Omega$. In proving their Schwarz lemma, Mok and Yau use the completeness of g only to apply Yau's Maximum Principle (see [15]). Because g is "complete near $\partial\Omega$", we may adapt the proof as follows.

Let $W = \bigcup_\alpha B'_\alpha$. Define $u(x) = \min_{\alpha, x \in B'_\alpha}(V_\alpha/V_g)$. The function u is positive and continuous on W. Since V_α and V_M are continuous on B_α, they are comparable on $B'_\alpha \cap \Omega$, so it suffices to show that $\overline{\lim}_{x \to \partial\Omega} u(x)$ is finite. Apply Lemma 1 with $X = (\Omega, g)$ and $f = -u^{-1/2n}$. If (b) of Lemma 1 is true, we are done; then $\overline{\lim}_{x \to \partial\Omega} u(x) \geq (u(q))^{-2n}$. We therefore suppose that (a) is true. From Mok and Yau's proof of the Schwarz lemma for volume forms we obtain

$$(2) \qquad \triangle(u^{-\frac{1}{2n}}) \leq \frac{-1}{2n}u^{-\frac{1}{2n}}(R + nu^{\frac{1}{n}}) + \frac{2n+1}{(2n)^2}u^{-\frac{1}{2n}-2}|\triangledown u|^2$$

for all points at which u is smooth. Here $\triangle$ denotes the Laplacian with respect to g and R the scalar curvature of g. Suppose u is smooth at an infinite number of the points q_k. Then using (a) and Yau's proof of his Maximum

Principle, we obtain a sequence $\{q_k'\} \subset W$ such that

$$(3) \qquad \lim_{k \to \infty} -(u(q_k'))^{-\frac{1}{2n}} = \overline{\lim_{x \to \partial \Omega}} -u(x)^{-\frac{1}{2n}},$$

$$(4) \qquad |\nabla (u(q_k'))^{-\frac{1}{2n}}| < \frac{1}{k},$$

$$(5) \qquad -\triangle (u(q_k'))^{-\frac{1}{2n}} < \frac{1}{k}.$$

Multiplying (2) by $u^{-\frac{1}{2n}}$ and applying (4) and (5) yields

$$-\frac{1}{k}(u(q_k'))^{-\frac{1}{2n}} < \frac{-1}{2n}(u(q_k'))^{-\frac{1}{n}}(R(q_k') + n(u(q_k'))^{\frac{1}{n}})) + \frac{2n+1}{k}.$$

Let $-c_1$ be a lower bound for R. Then

$$(6) \qquad -\frac{1}{k}(u(q_k'))^{-\frac{1}{2n}} < \frac{c_1}{2n}(u(q_k'))^{-\frac{1}{n}} - \frac{1}{2} + \frac{2n+1}{k}.$$

Equation (3) implies $\lim_{k \to \infty}(u(q_k')) = \overline{\lim}_{x \to \partial \Omega} u(x)$. We assume $\overline{\lim} u > 0$; if not, we are done. As $k \to \infty$, the left-hand side of (6) approaches 0 and the right-hand side approaches $\frac{c_1}{2n}(\overline{\lim} u)^{-\frac{1}{n}} - \frac{1}{2}$, so $\overline{\lim}_{x \to \partial \Omega} u(x) < c_1^n n^{-n}$.

If the points q_k must be chosen so that u is smooth at only a finite number of them, we still obtain a sequence $\{q_k'\}$ such that (3) holds, but we have neither the inequality (2) nor the estimates (4) and (5). Let S be the set of points in $\partial \Omega$ which are limit points of the points at which u is not smooth. Suppose that $\overline{\lim}_{x \to \partial \Omega} u(x)$ is infinite. If $y \in \partial \Omega \setminus S$, then $\overline{\lim}_{x \to y} u(x)$ is finite; otherwise, q_k' could be chosen in the set on which u is smooth. Now shift the balls B_α slightly to obtain a new covering $\{\tilde{B}_\alpha\}$ of $\partial \Omega$ so that the function $\tilde{u}$, which is constructed analogously to u, is smooth near all points of S. We obtain $\{\tilde{q}_k'\}$ as before. Because $\overline{\lim}_{x \to y} \tilde{u}(x)$ is infinite if and only if $\overline{\lim}_{x \to y} u(x)$ is infinite, we see that $\tilde{u}$ is smooth at an infinite number of the points $\tilde{q}_k'$. The previous argument then shows that $\overline{\lim}_{x \to \partial \Omega} \tilde{u}(x)$ is finite. Therefore, $\overline{\lim}_{x \to \partial \Omega} u(x)$ is finite.

Under the additional condition that V_M is defined on a neighborhood of $\bar{\Omega}$ and $(*)$ holds on all of Ω, the function $\tilde{\phi}$ is defined on all of Ω and is a continuous strictly plurisubharmonic exhaustion function for k large enough, so we conclude that Ω is Stein.

Remark. Mok and Yau used some similar techniques to show that given certain conditions on Ricci curvature, an increasing union of Stein subdomains in a Kähler manifold is Stein [6].

AN EXAMPLE OF GRAUERT

This theorem may be applied to an example of Grauert. In [4], he constructs several examples of domains with pseudoconvex boundary in a complex manifold which are not holomorphically convex. One of these is a pseudoconvex neighborhood U of the image of the zero section in a particular complex line bundle F over a complex manifold. In this example the total space F carries a Kähler metric with vanishing Ricci curvature (see [2]). The image of the zero section is a compact analytic set in U, so, by our earlier observations, U admits no Hermitian metric of strictly negative Ricci curvature, in particular, no Kähler-Einstein metric of negative Ricci curvature.

Grauert shows that U supports no nonconstant holomorphic functions. By Theorem 1, U admits no complete Kähler metric whose Ricci curvature is bounded below and whose Ricci tensor near ∂U is negative and bounded away from zero. In fact, in this particular example we may locally compare V_g to the volume form of a complete metric on a polydisc rather than to V_α, showing that $\log(V_g/V_F)$ is an exhaustion function. We therefore see that U may support no complete Kähler metric whose Ricci curvature is negative near ∂U and bounded below.

ACKNOWLEDGMENT

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