FULL SUBALGEBRAS OF JORDAN-BANACH ALGEBRAS AND ALGEBRA NORMS ON JB*-ALGEBRAS

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Abstract. We introduce normed Jordan Q-algebras, namely, normed Jordan algebras in which the set of quasi-invertible elements is open, and we prove that a normed Jordan algebra is a Q-algebra if and only if it is a full subalgebra of its completion. Homomorphisms from normed Jordan Q-algebras onto semisimple Jordan-Banach algebras with minimality of norm topology are continuous. As a consequence, the topology of the norm of a JB*-algebra is the smallest normable topology making the product continuous, and JB*-algebras have minimality of the norm. Some applications to (associative) C*-algebras are also given: (i) the associative normed algebras that are ranges of continuous (resp. contractive) Jordan homomorphisms from C*-algebras are bicontinuously (resp. isometrically) isomorphic to C*-algebras, and (ii) weakly compact Jordan homomorphisms from C*-algebras are of finite rank.

Introduction

Associative normed algebras in which the set of quasi-invertible elements is open were considered first by Kaplansky [15], who called them “normed Q-algebras”. Since then, normed Q-algebras were seldom studied (exceptions are Yood’s relevant papers [32, 33]) until the Wilansky conjecture [30], which states that associative normed Q-algebras are nothing but full subalgebras of Banach algebras. In fact, Palmer [20] set the bases for a systematic study of associative normed Q-algebras, providing, in particular, an affirmative answer to Wilansky’s conjecture (see also [3] for an independent proof of this result). It must also be mentioned that full subalgebras of Banach algebras have played a relevant role in connection with the nonassociative extension of Johnson’s uniqueness-of-norm theorem [25] and with the nonassociative extension of the Civin-Yood decomposition theorem [10].

The general theory of Jordan-Banach algebras began with the paper by Balachandran and Rema [2]; since then it has been fully developed in a complete analogy with the case of (associative) Banach algebras (see, e.g., [29, 16, 1, 9, 25, 10, 5, 11]), although in most of the cases new methods have been needed for such Jordan extensions of associative results. Noncomplete normed Jordan algebras whose sets of quasi-invertible elements are open (called, of course, “normed Jordan Q-algebras”) were only germinally considered in [29].
It is the aim of this paper to develop the theory of normed Jordan Q-algebras, providing also the complete analogy with the associative case. With no additional effort we shall even consider normed noncommutative Jordan Q-algebras so that the associative (or even alternative) case will remain contained in our approach. In the first part of this paper we give several characterizations of normed noncommutative Jordan Q-algebras (Theorem 4 and Proposition 6), including the one asserting that normed noncommutative Jordan Q-algebras are nothing but full subalgebras of noncommutative Jordan-Banach algebras (the affirmative answer to Wilansky's conjecture in the Jordan setting). It must also be emphasized that the normed complexification of a normed noncommutative Jordan real Q-algebra is also a Q-algebra (Proposition 3), whose proof needs an intrinsic Jordan method as it is the Shirshov-Cohn theorem with inverses [18]. We end this section with a theorem on automatic continuity (Theorem 8) which is a Jordan extension of the main result in [27].

The second part of the paper is devoted to applying a part of the developed theory of normed Jordan Q-algebras in order to obtain new results on JB*-algebras (hence on the Jordan structure of C*-algebras). Thus in Theorem 10 we use the aforementioned result on automatic continuity to generalize Cleveland's theorem [8], which asserts that the topology of the norm of a C*-algebra A is the smallest algebra-normable topology on A, to noncommutative JB*-algebras. (As a consequence, every norm on the vector space of a C*-algebra that makes the Jordan product continuous defines a topology which is stronger than the topology of the C*-norm—a result that improves the original Cleveland theorem.) The JB*-extension of Cleveland's result was obtained almost at the same time and with essentially identical techniques by Bensebah [4]. With the nonassociative Vidav-Palmer Theorem [24], it is also proved that noncommutative JB*-algebras have minimality of the norm (Proposition 11); i.e., |·| = ||·|| whenever |·| is any algebra norm satisfying |·| ≤ ||·||. Finally, with the main result in [26], we determine the associative normed algebras that are ranges of continuous Jordan homomorphisms from C*-algebras (Corollary 12), and we show that ranges of weakly compact Jordan homomorphisms from C*-algebras are finite dimensional (Corollary 13).

1. Preliminaries and notation

All the algebras we consider here are real or complex. A nonassociative algebra A satisfying x(yx) = (xy)x and x²(yx) = (x²y)x for all x, y in A is called a noncommutative Jordan (in short, n.c.J.) algebra. As usual A+ denotes the symmetrized algebra of A with product x • y = ½(xy + yx). Recall that A+ is a Jordan algebra whenever A is a n.c.J. algebra. For any element a in a n.c.J. algebra A, Ua denotes the linear operator on A defined by

\[ U_a(x) = a(ax + xa) - a²x = (ax + xa)a - xa², \quad x \in A. \]

Recall that Ua = Uₐ⁺, where Uₐ⁺ is the usual U-operator on the Jordan algebra A⁺. An element a in a n.c.J. algebra A with unit 1 is invertible with inverse b if ab = ba = 1 and a²b = ba² = a. This is equivalent to a being invertible with inverse b in the Jordan algebra A⁺ [19], whence, if Inv(A) denotes the set of invertible elements in A, then we have Inv(A) = Inv(A⁺). We recall the
following basic results (see [14, Theorem 13, p. 52]). For elements $x$, $y$ in $A$
- $x$ is invertible if and only if $U_x$ is an invertible operator, and in that case $x^{-1} = U_x^{-1}(x)$ and $U_x^{-1} = U_{x^{-1}}$.
- $x$ and $y$ are invertible if and only if $U_x(y)$ is invertible.
- $x$ is invertible if and only if $x^n$ ($n \geq 1$) is invertible

An element $a$ in a n.c.J. algebra $A$ is quasi-invertible with quasi-inverse $b$ if $1 - a$ has inverse $1 - b$ in the n.c.J. algebra $A_1$ (the unitization of $A$) obtained by adjoining a unit to $A$ in the usual way. Let $q - \text{Inv}(A)$ denote the set of quasi-invertible elements in $A$. Any real n.c.J. algebra $A$ can be regarded as a real subalgebra of a complex n.c.J. algebra $A_C$ which satisfies $A_C = A \oplus iA$ and is called the complexification of $A$ (see [6, Definition 3.1]). The spectrum of an element $x$ in a n.c.J. algebra $A$, denoted by $\text{sp}(x, A)$, is defined as in the associative case (see [6, Definitions 5.1 and 13.6]). The "algebraic" spectral radius of $x$ is defined by

$$
\rho(x, A) := \sup\{|\lambda|: \lambda \in \text{sp}(x, A)\}.
$$

We write $\text{sp}(x)$ or $\rho(x)$ instead of $\text{sp}(x, A)$ or $\rho(x, A)$ when no confusion can occur. A subalgebra $B$ of a n.c.J. algebra $A$ is called a full subalgebra of $A$ if $B$ contains the quasi inverses of its elements that are quasi-invertible in $A$, that is, the equality $q - \text{Inv}(B) = B \cap q - \text{Inv}(A)$ holds. Easy examples of full subalgebras are left or right ideals and strict inner ideals (see definition later). It is clear that if $B$ is a full subalgebra of a complex n.c.J. algebra $A$, then

$$
\text{sp}(x, A) \cup \{0\} = \text{sp}(x, B) \cup \{0\} \quad (x \in B).
$$

A n.c.J. algebra $A$ is said to be normed if an algebra norm (a norm $\| \cdot \|$ on the vector space of $A$ satisfying $\|ab\| \leq \|a\| \|b\|$ for all $a$, $b$ in $A$) is given on $A$. In that case the "geometric" spectral radius of $x \in A$ is the number $r_{\|\cdot\|}(x) := \lim \|x^n\|^{1/n}$. When no confusion can occur, we write $r(x)$ to denote $r_{\|\cdot\|}(x)$. The unitization $A_1$ of n.c.J. normed algebra $A$ becomes normed by defining $\|x + \alpha\| := \|x\| + \|\alpha\|$ for $x + \alpha$ in $A_1$. Also the complexification of a real n.c.J. normed algebra can be normed as in [6, Proposition 13.3]. Since every element $x$ in a n.c.J. normed algebra $A$ can be immersed in a closed associative full subalgebra of $A$ [5, Théorème 1], it follows that the properties of the spectrum and the classical functional calculus for a single element in (associative) Banach algebras remain valid for n.c.J. complete normed algebras. In particular, the Gelfand-Beurling formula, $r(x) = \rho(x)$, holds for any element $x$ in a n.c.J. complete normed algebra.

2. Noncommutative Jordan $Q$-algebras

A n.c.J. normed algebra $A$ in which the set $q - \text{Inv}(A)$ is open is called a n.c.J. $Q$-algebra. Taking into account that $A^+$ with the same norm as $A$ is a Jordan normed algebra and $q - \text{Inv}(A^+) = q - \text{Inv}(A)$, it is clear that $A$ is a n.c.J. $Q$-algebra if and only if $A^+$ is a Jordan $Q$-algebra. This fact will be used without comment in what follows. Also note that when $A$ has a unit, $q - \text{Inv}(A) = \{1 - x: x \in \text{Inv}(A)\}$, so $q - \text{Inv}(A)$ is open if and only if $\text{Inv}(A)$ is open. If $A$ is a n.c.J. complete normed algebra and $\phi$ denotes the mapping $x \mapsto U_{1-x}$ from $A$ into the Banach algebra $\text{BL}(A_1)$ of bounded linear operators on $A_1$, then $\phi$ is continuous and $q - \text{Inv}(A) = \phi^{-1}(\text{Inv}(\text{BL}(A_1)))$, so
q – Inv(A) is open and A is a n.c.J. Q-algebra. It is clear that full subalgebras of n.c.J. Q-algebras are also n.c.J. Q-algebras; in particular, full subalgebras of n.c.J. complete normed algebras are examples of n.c.J. Q-algebras. In fact, we shall prove in Theorem 4 that these examples exhaust the class of n.c.J. Q-algebras.

Proposition 1. If the set of quasi-invertible elements of a n.c.J. normed algebra A has some interior point, then A is a n.c.J. Q-algebra.

Proof. Suppose first that A has a unit and the set Inv(A) has some interior point, say x0. Choose y ∈ Inv(A). Then the linear operator Uy is a homeomorphism on A, and it leaves invariant the set Inv(A), so Uy(x0) is an interior point of Inv(A). Since the mapping z → Uz(x0), z ∈ A, is continuous, it follows that there is some number ρ > 0 such that Uz(x0) ∈ Inv(A), (hence, z ∈ Inv(A)) whenever ∥z − y∥ < ρ. Hence, Inv(A) is open.

Suppose now that the set q – Inv(A) has some interior point, say a0, and let ρ > 0 be such that u ∈ q – Inv(A) whenever ∥u0 − u∥ < ρ. Put δ = ρ/(1 + ρ + ∥u0∥) and x0 = 1 − u0 ∈ Inv(A1). Then for z = α + u in A1 such that ∥z − x0∥ < δ we have |1 − α| < δ and ∥αu + u∥ < δ + |1 + α||u∥, which implies α  0 and ∥u0 − (−u/α)∥ < δ/(1 + ∥u0∥)/(1 − δ) < ρ, so −u/α ∈ q − Inv(A); that is, z = α + u ∈ Inv(A1), and, as we noted in the beginning, this implies that the set Inv(A1) is open. Since A is an ideal of A1, it is also a full subalgebra of A1; hence, q − Inv(A) = A ∩ q − Inv(A1), which shows that the set q − Inv(A) is open. □

As a consequence of Proposition 1 and its proof we obtain

Proposition 2. Let A be a n.c.J. normed algebra and A1 its normed unitization. Then A is a n.c.J. Q-algebra if and only if the same is true for A1.

Proposition 3. Let A be a n.c.J. real normed algebra and Ac its normed complexification. Then A is a n.c.J. Q-algebra if and only if the same is true for Ac.

Proof. Assume that A is a real n.c.J. Q-algebra. We can suppose that A actually is a Jordan algebra and, by Proposition 2, that A has a unit. Let p denote the algebra norm on Ac defined as in [6, Proposition 13.3]. Choose 0 < α < 1 such that x ∈ Inv(A) whenever ∥1 − x∥ < α. Put δ = 3α. For a + ib ∈ Ac such that p(1 − (a + ib)) < δ we have max{∥1 − a∥, ∥b∥} < δ, so ∥1 − a∥ < δ < α; therefore, a ∈ Inv(A). Now

\[
\|1 - (a + U_b(a^{-1}))\| \leq \|1 - a\| + \|U_b(a^{-1})\| \leq \|1 - a\| + 3\|b\|^2\|a^{-1}\| \\
\leq \|1 - a\| + \frac{3\|b\|^2}{1 - \|1 - a\|} < \delta + \frac{3\delta^2}{1 - \delta} < \alpha ,
\]

which implies that a + U_b(a^{-1}) ∈ Inv(A). Next we shall prove

\[
U_{a+ib}(U_{a^{-1}}(a - ib)^2)) = (a + U_b(a^{-1}))^2.
\]

To this end note that if c = 1 + b then ∥1 − c∥ < α, so c ∈ Inv(A). Also note that the above equality can be localized to the subalgebra B of Ac generated by c, a, c^{-1}, and a^{-1}. By the Shirshov-Cohn theorem with inverses [18], B is a special Jordan algebra. Now in terms of the associative product of any associative envelop of B our equality is

\[
(a + ib)a^{-1}(a - ib)(a - ib)a^{-1}(a + ib) = (a + ba^{-1}b)^2 ,
\]
which can be easily verified. The equality just proved together with the fact that 
\[ a + U_{b}(a^{-1}) \in \text{Inv}(A) \subset \text{Inv}(A_{C}) \] 
gives that \( a + bi \in \text{Inv}(A_{C}) \). Hence the unit is an interior point of \( \text{Inv}(A_{C}) \) and, by Proposition 1, \( A_{C} \) is a n.c.J. \( Q \)-algebra. The converse is an easy consequence of the fact that \( A \) is a full real subalgebra of \( A_{C} \). □

**Theorem 4.** Let \( A \) be a n.c.J. normed algebra. The following are equivalent:

1. \( A \) is a n.c.J. \( Q \)-algebra.
2. \( \rho(x) = r(x) \) for all \( x \) in \( A \).
3. \( \rho(x) \leq \|x\| \) for all \( x \) in \( A \).
4. \( A \) is a full subalgebra of its normed completion.
5. \( A \) is a full subalgebra of some n.c.J. complete normed algebra.
6. Every element \( x \) in \( A \) with \( \|x\| < 1 \) is quasi-invertible in \( A \).

**Proof.** Suppose (i). Then there is some number \( \alpha > 0 \) such that \( x \in q - \text{Inv}(A) \) whenever \( \|x\| < \alpha \). By Propositions 2 and 3 we can assume that \( A \) is a complex Jordan \( Q \)-algebra with unit. Given \( x \) in \( A \) choose \( \lambda \in \mathbb{C} \) such that \( \|x\|/\alpha < |\lambda| \). Then \( \|x/\lambda\| < \alpha \), so \( 1 - x/\lambda \in \text{Inv}(A) \); that is, \( \lambda \notin \text{sp}(x) \). This shows that \( \rho(x) \leq \|x\|/\alpha \). Repeating with \( x \) replaced by \( x^n \) \((n \geq 1)\), we obtain \( \rho(x^n) \leq \|x^n\|/\alpha \). Since \( \text{sp}(x^n) = \{\lambda^n : \lambda \in \text{sp}(x)\} \) [16, Theorem 1.1] it follows that \( \rho(x^n) = \rho(x)^n \). Now taking \( n \)th roots in the above inequality and letting \( n \to \infty \), we see that \( \rho(x) \leq r(x) \). Now if \( \tilde{A} \) denotes the normed completion of \( A \), we have \( r(x) = \rho(x, \tilde{A}) \). Since \( \rho(x, \tilde{A}) < \rho(x) \), it follows that \( \rho(x) = r(x) = r(x, \tilde{A}) \), so (ii) is obtained. Clearly (ii) implies (iii). Next suppose (iii). Since for \( z = a + x \) in \( A_1 \) we have \( \rho(z, A_1) \leq \rho(x) + |a| \), (iii) is valid for both \( A \) and \( A_1 \), so we can assume that \( A \) has a unit. Let \( \tilde{A} \) denote the normed completion of \( A \) and choose \( a \in A \cap \text{Inv}(\tilde{A}) \). Then \( U_a \) is a linear homeomorphism on \( \tilde{A} \), and, in particular, \( U_a(A) \) is dense in \( A \). Therefore, there is \( b \in A \) such that \( \|1 - U_a(b)\| < 1 \); whence, \( \rho(1 - U_a(b)) < 1 \). From this we have \( U_a(b) \in \text{Inv}(A) \), which implies that \( a \in \text{Inv}(A) \). We have proved that \( A \cap \text{Inv}(\tilde{A}) \subset \text{Inv}(A) \). Since the opposite inclusion is always true, we have \( A \cap \text{Inv}(\tilde{A}) = \text{Inv}(A) \) and (iv) follows. Clearly (iv) implies (v). Suppose now that \( A \) is a full subalgebra of a n.c.J. complete normed algebra \( J \). Then \( x \in q - \text{Inv}(J) \) whenever \( x \in J \) with \( \|x\| < 1 \), because \( J \) is complete. In particular, if \( x \in A \) and \( \|x\| < 1 \), then \( x \in A \cap q - \text{Inv}(J) = q - \text{Inv}(A) \) and (vi) follows. Finally, by Proposition 1, (vi) implies (i). □

As a clear consequence of (v) the spectrum of an element in a n.c.J. \( Q \)-algebra is a compact (nonempty) subset of \( \mathbb{C} \). For associative \( Q \)-algebras the equivalence of (i), (ii), and (iii) of Theorem 4 was proved by Yood [32, Lemma 2.1]. Also Palmer in [20, Theorem 3.1 and Proposition 5.10] states the associative version of Theorem 4. Next we are going to give a characterization of n.c.J. \( Q \)-algebras as those n.c.J. normed algebras in which the maximal modular inner ideals are closed.

A vector subspace \( M \) of a Jordan algebra \( A \) such that \( U_m(A) \subset M \) for all \( m \in M \) is called an inner ideal of \( A \). If, in addition, \( M \) is also a subalgebra of \( A \), then it is called a strict inner ideal of \( A \). Recall that for \( a, b \) in \( A \) the operator \( U_{a,b} \) is defined by \( U_{a,b} = (U_{a+b} - U_{a} - U_{b})/2 \). The element \( U_{a,b}(x) \) is usually written as \( \{a, x, b\} \). A strict inner ideal \( M \) of \( A \) is called
$x$-modular for some $x \in A$ when the following three conditions are satisfied:

(i) $U_{1-x}(A) \subset M$.

(ii) $\{1-x, z, m\} \in M$ for all $z \in A_1$ and all $m \in M$.

(iii) $x^2 - x \in M$.

This concept of modularity in Jordan algebras is due to Hogben and McCrimmon [13]. The next result has been used in [9], giving the clue for its proof in the case of Jordan-Banach algebras, although it has not been explicitly stated.

**Proposition 5.** The closure $\overline{M}$ of a proper $x$-modular strict inner ideal $M$ of a Jordan $Q$-algebra $A$ is a proper $x$-modular strict inner ideal of $A$.

**Proof.** Using the continuity of the product of $A$, it is easily obtained that $\overline{M}$ is an $x$-modular strict inner ideal of $A$. Let us show it is proper. Choose $m \in M$, and let $z = x - m$. If $||z|| < 1$, then by Theorem 4 we know that $z \in q - \text{Inv}(A)$. If $w$ is the quasi inverse of $z$, then $1 - z = U_{1-z}(1 - w) = U_{1-z}(1 - w)^2 - U_{1-z}(w^2 - w) = 1 - U_{1-z}(w^2 - w)$, so $z = U_{1-z}(t)$, where $t = w^2 - w \in A$. Now

$$z = U_{1-z}(t) = U_{1-x-m}(t) = U_{1-x}(t) + U_{m}(t) + 2U_{1-x,m}(t),$$

and it follows that $z \in M$, but then $x \in M$, and this implies that $M = A$ [13, Proposition 3.1], which contradicts the assumption that $M$ is proper. Hence it must be $||x - m|| \geq 1$ for every $m \in M$, so $x \notin \overline{M}$. Thus $\overline{M}$ is proper. □

A maximal modular inner ideal of a Jordan algebra $A$ is a strict inner ideal which is $x$-modular for some $x \in A$ and maximal among all proper $x$-modular strict inner ideals of $A$ (for $x$ fixed). The maximal modular inner ideals of a n.c.J. algebra $A$ are, by definition, the maximal modular inner ideals of the Jordan algebra $A^+$. 

**Proposition 6.** Let $A$ be a n.c.J. normed algebra. The following are equivalent:

(i) $A$ is a n.c.J. $Q$-algebra.

(ii) The maximal modular inner ideals of $A$ are closed.

**Proof.** As a consequence of Proposition 5 we have that (i) implies (ii). To prove the converse we can suppose that $A$ is a Jordan algebra. Let $\hat{A}$ denote the normed completion of $A$. Choose $x \in A \cap q - \text{Inv}(A)$. Then $1 - x$ is invertible in $\hat{A}_1$, so $U_{1-x}$ is a homeomorphism on $\hat{A}_1$; in particular, $U_{1-x}(A_1)$ is dense in $A_1$. Therefore, if $z \in A$, there is a sequence $\{\alpha_n + z_n\}$ in $A_1$ such that $\lim U_{1-x}(\alpha_n + z_n) = z$. Since $U_{1-x}(\alpha_n + z_n)$ can be written in the form $\alpha_n + w_n$ with $w_n \in A$, it follows that $\lim \{\alpha_n\} = 0$, and we deduce that $\lim \{U_{1-x}(z_n)\} = z$. Hence $U_{1-x}(A)$ is dense in $A$. Note that $U_{1-x}(A) \subset A$, since $A$ is an ideal of $A_1$. If $U_{1-x}(A) \neq A$, then it follows from [13, Remark 2.8] that there is a maximal modular inner ideal $M$ of $A$ such that $U_{1-x}(A) \subset M$. Since, by assumption, $M$ is closed, we have a contradiction with the density of $U_{1-x}(A)$ in $A$. Hence $U_{1-x}(A) = A$. It has been seen in the proof of Proposition 5 that the quasi inverse $y$ of $x$ is given by $y = U_{1-y}(x^2 - x) = U_{1-x}^{-1}(x^2 - x)$, so it follows that $y$ lies in $A$. We have proved that $A$ is a full subalgebra of $\hat{A}$, and therefore $A$ is a Jordan $Q$-algebra. □

The maximal modular left or right ideals in associative algebras are also maximal modular inner ideals [13, Example 3.3]. In this respect the above
proof can be easily modified to show that, if $A$ is a normed associative algebra and the maximal modular left ideals of $A$ are closed, then $A$ is an associative $Q$-algebra (see also [33, Theorem 2.9]).

Since for any element $x$ in a n.c.J. $Q$-algebra we have $\rho(x) = r(x)$, it follows that homomorphisms of n.c.J. $Q$-algebras decrease the (geometric) spectral radius. Moreover, if $r(x) = 0$ then $sp(x) = \{0\}$, so $x$ is quasi-invertible. Taking this into account, it is easily seen that the proof given by Aupetit [1] and the recent and more simple proof given by Ransford [23] of Johnson's uniqueness of norm theorem yield immediately to the following result (see also [25, Proposition 3.1]). If $X$ and $Y$ are normed spaces and $F$ is a linear mapping from $X$ into $Y$, we denote by $S(F)$ (the separating subspace of $F$) the set of those $y$ in $Y$ for which there is a sequence $\{x_n\}$ in $X$ such that $lim\{x_n\} = 0$ and $lim\{F(x_n)\} = y$. If $A$ is a n.c.J. algebra, $Rad(A)$ means the Jacobson radical of $A$ [19]; namely, $Rad(A)$ is the largest quasi-invertible ideal of $A$. If $Rad(A) = \{0\}$, $A$ is called semisimple.

**Proposition 7** [1, 23]. Let $A$ and $B$ be n.c.J. complex $Q$-algebras, and let $F$ be a homomorphism from $A$ into $B$. Then $r(b) = 0$ for every $b$ in $S(F) \cap F(A)$. Moreover, if $F$ is a surjective homomorphism, then $S(F) \subset Rad(B)$.

Suppose $A$ is a n.c.J. $Q$-algebra, and let $M$ be a closed ideal of $A$. Then the algebra $A/M$ is a n.c.J. $Q$-algebra. (Indeed, if $\pi$ denotes the canonical projection of $A$ onto $A/M$, then $\pi$ is open and $\pi(q - Inv(A)) \subset q - Inv(A/M)$. Hence we may apply Proposition 1 to $A/M$.) Moreover, if $B$ is a semisimple n.c.J. algebra and $\varphi$ is a homomorphism from $A$ onto $B$, then $Ker(\varphi)$ is closed (just use Theorem 4(vi) to obtain in the usual way that $\varphi(Ker(\varphi))$ is a quasi-invertible ideal of $B$). With Proposition 7 and these considerations the proof of the main result in [27] yields directly to the following result. Recall that a normed algebra $(A, \| \cdot \|)$ is said to have minimality of norm topology if any algebra norm on $A$, $| \cdot |$, minorizing $\| \cdot \|$ and $| \cdot | \leq \alpha \| \cdot \|$ for some $\alpha > 0$, is actually equivalent to $\| \cdot \|$.

**Theorem 8.** Let $A$ be a n.c.J. complex $Q$-algebra, and let $B$ be a semisimple complete normed complex n.c.J. algebra with minimality of norm topology. Then every homomorphism from $A$ onto $B$ is continuous.

### 3. Algebra norms on noncommutative JB*-algebras

A not necessarily commutative (for short n.c.) $JB^*$-algebra $A$ is a complete normed n.c.J. complex algebra with (conjugate linear) algebra involution * such that $\|U_{a}(a^{*})\| = \|a\|^3$ for all $a$ in $A$. Thus $C^*$-algebras and (commutative) $JB^*$-algebras are particular types of n.c. $JB^*$-algebras. If $A$ is a n.c. $JB^*$-algebra, then $A^{+}$ is a $JB^*$-algebra with the same norm and involution as those of $A$. $JB^*$-algebras were introduced by Kaplansky in 1976, and they have been extensively studied since the paper by Wright [31].

**Lemma 9.** If $| \cdot |$ is any algebra norm on a n.c. $JB^*$-algebra $A$, then $(A, | \cdot |)$ is a n.c.J. $Q$-algebra.

**Proof.** Since n.c.J. algebras are power-associative, the closed subalgebra of $A$ generated by a symmetric element $(a = a^*)$ is a commutative $C^*$-algebra. Given $a$ in $A$, we can consider the commutative $C^*$-algebra generated by the
symmetric element \( a^* \cdot a = \frac{1}{2}(aa^* + a^*a) \) and make use of a well-known result due to Kaplansky, according to which any algebra norm on a commutative \( C^* \)-algebra is greater than the original norm, to get that \( \|a^* \cdot a\| \leq \|a^* \cdot a\| \). Also it is known that \( \|a\|^2 \leq 2\|a^* \cdot a\| \) [21, Proposition 2.2]. So we have that \( \|a\|^2 \leq 2|a^* \cdot a| \leq 2|a^*||a| \) for all \( a \) in \( A \). Hence \( \|a^n\|^2 \leq 2\|(a^*)^n\||a^*|^n \) for all \( n \) in \( \mathbb{N} \), which implies that \( (\|a\|^2)^n \leq r_{\|\cdot\|}(a^*)^n \|a^*\|^n \). Now, if \( (C, \|\cdot\|) \) denotes the completion of \( (A, \|\cdot\|) \), we have \( r_{\|\cdot\|}(a) = \rho(a, C) \leq \rho(a, A) = r_{\|\cdot\|}(a) \) for all \( a \) in \( A \). Thus \( (\|a\|^2)^n \leq r_{\|\cdot\|}(a^*)^n r_{\|\cdot\|}(a) \) and consequently \( r_{\|\cdot\|}(a) \leq r_{\|\cdot\|}(a) \). We deduce that \( r_{\|\cdot\|}(a) = r_{\|\cdot\|}(a) = \rho(a, A) \) for all \( a \) in \( A \), and by Theorem 4 we conclude that \( (A, \|\cdot\|) \) is a n.c. \( Q \)-algebra. \( \square \)

**Theorem 10.** The topology of the norm of a n.c. \( JB^* \)-algebra \( A \) is the smallest algebra normable topology on \( A \).

**Proof.** If \( |\cdot| \) is any algebra norm on \( A \), it has been shown in the proof of Lemma 9 that \( \|a\|^2 \leq 2|a^*||a| \) for all \( a \) in \( A \). If we know additionally that \( |\cdot| \leq M\|\cdot\| \) for some nonnegative number \( M \), then \( \|a\|^2 \leq 2M\|a^*\||a| = 2M\|a\||a| \), so \( \|a\| \leq 2M\|a| \) for all \( a \) in \( A \). Hence the norm \( |\cdot| \) is equivalent to the norm of \( A \). Therefore, \( (A, \|\cdot\|) \) has minimality of norm topology. Now, for an arbitrary algebra norm \( |\cdot| \) on \( A \), we can use Lemma 9 and apply Theorem 8 to the identity mapping from \( (A, |\cdot|) \) into \( (A, \|\cdot\|) \) to obtain that this mapping is continuous, which concludes the proof. \( \square \)

If \( A \) is a \( C^* \)-algebra, then the particularization of Theorem 10 to the \( JB^* \)-algebra \( A^+ \) gives that any algebra norm on \( A^+ \) defines a topology on \( A \) which is stronger than the original one. This is an improvement of the classical result by Cleveland [8] which states the same for algebra norms on \( A \).

Unlike the preceeding results, which are of an algebraic-topologic nature, the following one is geometric.

Let \( A \) be a complete normed complex nonassociative algebra with unit \( 1 \) such that \( \|1\| = 1 \). Denote by \( A^* \) the dual Banach space of \( A \). For \( a \) in \( A \) the subset of \( \mathbb{C} \), \( V_{\|\cdot\|}(a) = \{f(a) : f \in A^*, \|f\| = 1 = f(1)\} \) is called the numerical range of \( a \). The set of hermitian elements of \( A \), denoted by \( H(A) \), is defined as the set of those elements \( a \) in \( A \) such that \( V_{\|\cdot\|}(a) \subset \mathbb{R} \). If \( A = H(A) + iH(A) \), then \( A \) is called a \( V \)-algebra. The general nonassociative Vidav-Palmer theorem [24] says that the class of (nonassociative) \( V \)-algebras coincides with the one of unital n.c. \( JB^* \)-algebras.

**Proposition 11.** Every n.c. \( JB^* \)-algebra \( A \) has the property of minimality of the norm; that is, if \( |\cdot| \) is an algebra norm on \( A \) such that \( |\cdot| \leq \|\cdot\| \), then the equality \( |\cdot| = \|\cdot\| \) holds.

**Proof.** By Theorem 10 and the assumptions made, \(|\cdot|\) and \(\|\cdot\|\) are equivalent norms on \( A \), so \(|\cdot|\) is a complete norm on \( A \). Suppose first that \( A \) has a unit element \( 1 \). \(|\cdot|\) being an algebra norm, we have \( 1 = |1| \leq \|1\| = 1 \), so \(|1| = 1 \). Let \(\|\cdot\|\) and \(|\cdot|\) also denote the corresponding dual norms of \(\|\cdot\|\) and \(|\cdot|\). Then for \( f \) in \( A^* \) we have \( \|f\| \leq |f| \), and we deduce easily that \( V_{|\cdot|}(a) \subset V_{\|\cdot\|}(a) \) for all \( a \) in \( A \). Since \( (A, \|\cdot\|) \) is a \( V \)-algebra, it follows that \( (A, |\cdot|) \) is also a \( V \)-algebra, and, consequently, by the nonassociative Vidav-Palmer theorem, \( (A, |\cdot|) \) is a n.c. \( JB^* \)-algebra. Since the norm of a n.c. \( JB^* \)-algebra is unique [31], we conclude that \(|\cdot| = \|\cdot\|\). If \( A \) has no unit element, then it is known that \( (A^{**}, \|\cdot\|) \), with the Aren’s product and a convenient involution which
extends that of $A$, is a unital n.c. $JB^*$-algebra [21]. Since the bidual $A^{**}$ of $A$ is the same for both norms and $|\cdot|$ is an algebra norm on $A^{**}$ satisfying $|\cdot| \leq \|\cdot\|$ on $A^{**}$, it follows from what was previously seen that $|\cdot| = \|\cdot\|$ on $A^{**}$ and, in particular, $|\cdot| = \|\cdot\|$ on $A$. \qed

Now we apply Theorem 10 and Proposition 11 to the study of the ranges of Jordan homomorphisms from $C^*$-algebras.

**Corollary 12.** Assume that a normed associative complex algebra $B$ is the range of a continuous (resp. contractive) Jordan homomorphism from a $C^*$-algebra. Then $B$ is bicontinuously (resp. isometrically) isomorphic to a $C^*$-algebra.

**Proof.** Let $A$ be a $C^*$-algebra and $\varphi$ a Jordan homomorphism from $A$ onto $B$ under the assumptions in the statement. Since closed Jordan ideals of a $C^*$-algebra are associative ideals (see [7, Theorem 5.3.] or [21, Theorem 4.3]), $A/\text{Ker}(\varphi)$ is a $C^*$-algebra and we may assume that $\varphi$ is a one-to-one mapping. Then, by Theorem 10 (resp. Proposition 11) applied to the $JB^*$-algebra $A^+$, it follows that $\varphi$ is a bicontinuous (resp. isometric) Jordan isomorphism from $A$ onto $B$. Let $C$ denote the associative complex algebra with vector space that of $A$ and product $\Box$ defined by $x \Box y := \varphi^{-1}(\varphi(x)\varphi(y))$. Then $C^+ (= A^+)$ is a $JB^*$-algebra under the norm and involution of $A$, so, with the same norm and involution, $C$ becomes a $C^*$-algebra [26, Theorem 2] and, clearly, $\varphi$ becomes a bicontinuous (resp. isometric) associative isomorphism from $C$ onto $B$. \qed

**Corollary 13.** The range of any weakly compact Jordan homomorphism from a $C^*$-algebra into a normed algebra is finite dimensional.

**Proof.** If $A$ is a $C^*$-algebra, $B$ a normed algebra, and $\varphi$ a weakly compact Jordan homomorphism from $A$ into $B$, then, as above, $A/\text{Ker}(\varphi)$ is a $C^*$-algebra and, easily, the induced Jordan homomorphism $A/\text{Ker}(\varphi) \rightarrow B$ is weakly compact, so again we may assume that $\varphi$ is a one-to-one mapping. Now, by Theorem 10 applied to $A^+$, $\varphi$ is a weakly compact topological embedding, so $A$ is a $C^*$-algebra with reflexive Banach space, and so $A$ (and hence the range of $\varphi$) is finite dimensional [28]. \qed

**Remark 14.** The fact that weakly compact (associative) homomorphisms from $C^*$-algebras have finite-dimensional ranges was proved first in [12] as a consequence of a more general result, and later a very simple proof (that we imitate above) was obtained by Mathieu [17]. If $A$ is a n.c. $JB^*$-algebra and $\varphi$ is any weakly compact homomorphism from $A$ into a normed algebra $B$, since $A/\text{Ker}(\varphi)$ is a n.c. $JB^*$-algebra [21, Corollary 1.11], to obtain some information about the range of $\varphi$ we may assume that $\varphi$ is a one-to-one mapping, and then, as above, the range of $\varphi$ is bicontinuously isomorphic to a n.c. $JB^*$-algebra with reflexive Banach space, namely, a finite product of simple n.c. $JB^*$-algebras which are either finite dimensional or quadratic [22, Theorem 3.5] (note that infinite-dimensional quadratic $JB^*$-algebras do exist and the identity mapping on such a $JB^*$-algebra is weakly compact). This result on the range of a weakly compact homomorphism from a n.c. $JB^*$-algebra was proved first in [11] by using Theorem 10 and a nonassociative extension of the above-mentioned general result in [12]. The proof given above (also suggested in [11]) is analogous to Mathieu's proof for the particular case of $C^*$-algebras.
References


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