NONEXISTENCE OF WEAKLY ALMOST COMPLEX STRUCTURES
ON GRASSMANNIANS

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Abstract. In this paper we prove that, for 2 ≤ k ≤ n/2, the unoriented Grassmann manifold G_k(\mathbb{R}^n) admits a weakly almost complex structure if and only if n = 2k = 4 or 6; for 3 ≤ k ≤ \frac{n}{2}, none of the oriented Grassmann manifolds G_k(\mathbb{R}^n) except G_3(\mathbb{R}^6) and a few as yet undecided ones—admits a weakly almost complex structure.

1. Introduction

For 1 ≤ k < n, let \tilde{G}_k(\mathbb{R}^n) (G_k(\mathbb{R}^n) resp.) denote the oriented (unoriented) Grassmann manifold of the oriented (unoriented) k-dimensional vector subspace of \mathbb{R}^n. \tilde{G}_k(\mathbb{R}^n) (G_k(\mathbb{R}^n)) is a smooth manifold of dimension k(n - k). Note that \tilde{G}_1(\mathbb{R}^n) \cong S^{n-1}(G_1(\mathbb{R}^n) \cong RP^{n-1}), the (n-1)-sphere (real projective space), and that \tilde{G}_k(\mathbb{R}^n) \cong G_{n-k}(\mathbb{R}^n) (G_k(\mathbb{R}^n) \cong G_{n-k}(\mathbb{R}^n)) under the diffeomorphism that sends a k-plane V to its orthogonal complement V⊥.

Recall that a smooth manifold M is said to be (weakly) almost complex if its tangent bundle τM is (stably) isomorphic to the realification of a complex vector bundle over M.

For example, \tilde{G}_1(\mathbb{R}^n) \cong S^{n-1} is weakly almost complex for all n but is almost complex only when n = 3 or 7 \cite{1}; G_1(\mathbb{R}^n) \cong RP^{n-1} is weakly almost complex only when n is even. It is a classical result that G_2(\mathbb{R}^n) \cong SO(n)/(SO(2) \times SO(n-2)) is a Hermitian symmetric space and is therefore almost complex for all n. Our main results are:

Theorem 1.1. Let 2 ≤ k ≤ \frac{n}{2}. Then G_k(\mathbb{R}^n) is weakly almost complex if and only if n = 2k = 4 or 6.

Theorem 1.2. Let 3 ≤ k ≤ \frac{n}{2}. Then \tilde{G}_k(\mathbb{R}^n) is not weakly almost complex if n is odd or (n-k) ≥ 8.

Our results are sharper than that in \cite{6}. Note that \tilde{G}_3(\mathbb{R}^6) is weakly almost complex \cite{6}. The unsolved cases for weak complexity of \tilde{G}_k(\mathbb{R}^n) are \tilde{G}_4(\mathbb{R}^8), \tilde{G}_5(\mathbb{R}^{10}), \tilde{G}_6(\mathbb{R}^{12}), \tilde{G}_7(\mathbb{R}^{14}), \tilde{G}_3(\mathbb{R}^8), \tilde{G}_4(\mathbb{R}^{10}), \tilde{G}_5(\mathbb{R}^{12}), and \tilde{G}_3(\mathbb{R}^{10}).
\( \gamma_{n,k} \) denote the canonical \( k \)-plane bundle over \( \tilde{G}_k(\mathbb{R}^n) \) \((G_k(\mathbb{R}^n))\), and let \( \beta_{n,k} \) \((\beta_{n,k})\) be its orthogonal complement, whose fiber over a \( V \in \tilde{G}_k(\mathbb{R}^n) \) \((G_k(\mathbb{R}^n))\) is the vector space \( V^\perp \subset \mathbb{R}^n \). We have bundle equivalence
\[
\gamma_{n,k} \oplus \beta_{n,k} \cong ne \quad (\gamma_{n,k} \oplus \beta_{n,k} \cong ne),
\]
where \( e \) denotes a trivial line bundle.

It is well known that the tangent bundle \( \tau \tilde{G}_k(\mathbb{R}^n) \) \((\tau G_k(\mathbb{R}^n))\) of \( \tilde{G}_k(\mathbb{R}^n) \) has the following description (see [4]):
\[
\tau \tilde{G}_k(\mathbb{R}^n) \cong \gamma_{n,k} \otimes \beta_{n,k} \quad (\tau G_k(\mathbb{R}^n) \cong \gamma_{n,k} \otimes \beta_{n,k}).
\]
Using (1.3) and (1.4), we obtain
\[
\tau \tilde{G}_k(\mathbb{R}^n) \oplus (\gamma_{n,k} \otimes \gamma_{n,k}) \cong n\gamma_{n,k} \quad (\tau G_k(\mathbb{R}^n) \oplus (\gamma_{n,k} \otimes \gamma_{n,k}) \cong n\gamma_{n,k}).
\]

For a CW complex \( X \), let \( r: K(X) \rightarrow KO(X) \) denote the homomorphism of Abelian groups gotten by restriction of scalars to \( \mathbb{R} \), and let \( c: KO(X) \rightarrow K(X) \) denote the complexification, \( c[\xi] = [\xi \otimes_{\mathbb{R}} \mathbb{C}] \), which is a ring homomorphism.

We have the following identity:
\[
rc(x) = 2x \quad \forall x \in KO(X).
\]

### 2. The unoriented Grassmannians

**Lemma 2.1.** \( G_2(\mathbb{R}^6) \) is not weakly almost complex.

**Proof.** It is well known that
\[
H^*(G_2(\mathbb{R}^6); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \overline{w_1}, \overline{w_2}, \overline{w_3}, \overline{w_4}]
\]
modulo the relation \((1 + w_1 + w_2)(1 + \overline{w_1} + \overline{w_2} + \overline{w_3} + \overline{w_4}) = 1\), so
\[
H^*(G_2(\mathbb{R}^6); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2]/\langle w_1^5 + w_1 w_2^2 + w_1 w_3 w_2 + w_2^3\rangle.
\]
The fact \( H^8(G_2(\mathbb{R}^6); \mathbb{Z}_2) \cong \mathbb{Z}_2 \) implies \( w_2^4 \neq 0 \). By (1.5), the total Stiefel-Whitney classes of \( G_2(\mathbb{R}^6) \) are given by
\[
w(G_2(\mathbb{R}^6)) = (1 + w_1 + w_2)^6/(1 + w_2^2) = 1 + (w_1^4 + w_2^2) + w_1^2 w_2^2 + w_2^4.
\]
This gives
\[
w_2(G_2(\mathbb{R}^6)) = 0, \quad w_8(G_2(\mathbb{R}^6)) = w_2^4 \neq 0.
\]
The following results follow immediately from Wu's formula \( sq^1 w_2 = w_1 w_2 \)[5]:
\[
sq(w_1^6) = w_1^6, \quad sq(w_1^4 w_2) = w_1^4 w_2 + w_1^5 w_2,

sq(w_1^2 w_2^2) = w_1^2 w_2^2, \quad sq(w_2^3) = w_2^3 + w_1^2 w_2 w_1.
\]
Therefore, \( sq: H^6(G_2(\mathbb{R}^6); \mathbb{Z}_2) \rightarrow H^8(G_2(\mathbb{R}^6); \mathbb{Z}_2) \) is zero. Hence, \( w_8(G_2(\mathbb{R}^6)) \) is not in the image of \( H^6(G_2(\mathbb{R}^6); \mathbb{Z}) \) under the homomorphism \( sq^2 \). Our lemma immediately follows from the following criterion [3]: \( M^8 \) admits a weakly almost complex structure iff \( \delta w_2(M) = 0 \) and \( w_8(M) \in sq^2 H^6(M; \mathbb{Z}) \).

**Lemma 2.2.** If \( G_k(\mathbb{R}^n) \) is weakly almost complex, then so are \( G_{k-1}(\mathbb{R}^{n-2}) \) and \( G_k(\mathbb{R}^{n-2}) \).

**Proof.** Let us consider the maps
\[
G_{k-1}(\mathbb{R}^{n-2}) \overset{j}{\rightarrow} G_{k-1}(\mathbb{R}^{n-1}) \overset{j}{\rightarrow} G_k(\mathbb{R}^n)
\]
where \( i \) regards a \( V \) in \( \mathbb{R}^{n-2} \) as a \( V \) in \( \mathbb{R}^{n-1} \) and \( j \) sends a \( V \) to \( V \oplus \mathbb{R} \). It is easy to see that

\[
\begin{align*}
(i^* \gamma_{n-1,k-1}) & \cong \gamma_{n-2,k-1}, & (i^* \beta_{n-1,k-1}) & \cong \beta_{n-2,k-1} \oplus \epsilon \\
(j^* \gamma_{n,k}) & \cong \gamma_{n-1,k-1} \oplus \epsilon, & (j^* \beta_{n,k}) & \cong \beta_{n-1,k-1}.
\end{align*}
\]

So we have

\[
(j \circ i)^* \tau G_k(\mathbb{R}^n) \cong i^* \circ j^* (\gamma_{n,k} \otimes \beta_{n,k}) \\
\cong i^* (\gamma_{n-1,k-1} \oplus \epsilon) \otimes i^* (\beta_{n-1,k-1}) \\
\cong (\gamma_{n-2,k-1} \oplus \epsilon) \otimes (\beta_{n-2,k-1} \oplus \epsilon) \\
\cong \gamma_{n-2,k-1} \oplus \beta_{n-2,k-1} \oplus \gamma_{n-2,k-1} \oplus \beta_{n-2,k-1} \oplus \epsilon \\
\cong \tau G_{k-1}(\mathbb{R}^{n-2}) \oplus (n-1)\epsilon.
\]

So the conclusion for \( G_{k-1}(\mathbb{R}^{n-2}) \) is true.

Let us consider the maps

\[
G_k(\mathbb{R}^{n-2}) \xrightarrow{i} G_k(\mathbb{R}^{n-1}) \xrightarrow{i} G_k(\mathbb{R}^n).
\]

By (2.3), we obtain

\[
(i_2 \circ i_1)^* \tau G_k(\mathbb{R}^n) \cong i_1^* \circ i_2^* (\gamma_{n,k} \otimes \beta_{n,k}) \cong i_1^* (\gamma_{n-1,k} \otimes (i_1^* (\beta_{n-1,k} \oplus \epsilon)) \\
\cong \gamma_{n-2,k} \otimes (\beta_{n-2,k} \oplus \epsilon \oplus \epsilon) \cong \tau G_k(\mathbb{R}^{n-2}) \oplus 2\gamma_{n-2,k}.
\]

By (1.6), \( 2\gamma_{n-2,k} \) is in the image of \( r: K(G_k(\mathbb{R}^{n-2})) \to KO(G_k(\mathbb{R}^{n-2})) \). This completes the proof.

**Proof of Theorem 1.1.** The statement that \( G_2(\mathbb{R}^4) \) and \( G_3(\mathbb{R}^6) \) are weakly almost complex was obtained in [6].

We note that \( G_2(\mathbb{R}^{2n+1}) \) is not weakly almost complex, since it is not orientable. The "only if" part of the theorem may be shown by using this fact, Lemma 2.1, and Lemma 2.2 repeatedly.

**Remark.** Borel and Hirzebruch [2, p. 526] proved that \( G_2(\mathbb{R}^n) \) is not almost complex if \( n \geq 5 \). We extend their results.

### 3. The oriented Grassmannians

**Proof of Theorem 1.2.** If \( n \) is odd, \( 3 \leq k \leq n/2 \), then \( \tilde{G}_k(\mathbb{R}^n) \) is not weakly almost complex. The reason is that \( w_3(\tilde{G}_k(\mathbb{R}^n)) \neq 0 \) [6].

By Lemma 2.1, \( G_2(\mathbb{R}^6) \) is not weakly almost complex. But \( \tau G_2(\mathbb{R}^6) \oplus (\gamma_{6,2} \otimes \gamma_{6,2}) \cong 6\gamma_{6,2} \). So we see that the element \( \gamma_{6,2} \otimes \gamma_{6,2} \) is not in the image of \( r: K(G_2(\mathbb{R}^6)) \to KO(G_2(\mathbb{R}^6)) \).

Let \( \xi \) denote the line bundle whose \( w_1(\xi) \) equals \( w_1(\gamma_{6,2}) \). Then \( \xi \otimes \gamma_{6,2} \) is an orientable 3-plane bundle with

\[
(\xi \otimes \gamma_{6,2}) \otimes (\xi \otimes \gamma_{6,2}) \cong \gamma_{6,2} \otimes \gamma_{6,2} \otimes 2\gamma_{6,2} \otimes \xi \otimes \epsilon.
\]

Then we have

\[
(\xi \otimes \gamma_{6,2})^2 \otimes \epsilon \notin \text{Im } r.
\]

Now let \( n \) be even, \( k \geq 3 \), and \( n-k \geq 8 = \dim G_2(\mathbb{R}^6) \). Since \( \tilde{G}_k(\mathbb{R}^n) \) is \((n-k)\)-universal for orientable \( k \)-plane bundles, there exists a map \( f: G_2(\mathbb{R}^6) \to \)
such that \( f^*(\tilde{\gamma}_{n,k}) \cong \xi \oplus \gamma_{6,2} \oplus me \), where \( m = k - 3 \). We have

\[
\begin{align*}
& f^*(\tilde{\gamma}_{n,k} \otimes \tilde{\gamma}_{n,k}) \cong (\xi \oplus \gamma_{6,2})^2 \oplus m^2e \oplus 2m(\xi \oplus \gamma_{6,2}), \\
& f^*\tau\tilde{G}_k(\mathbb{R}^n) \otimes (\xi \oplus \gamma_{6,2})^2 \oplus m^2e \oplus 2m(\xi \oplus \gamma_{6,2}) \cong n f^*(\tilde{\gamma}_{n,k}).
\end{align*}
\]

Using (3.1), (1.6), and the fact that \( n \) is even, we see that \( \tilde{G}_k(\mathbb{R}^n) \) is not weakly almost complex. This completes the proof of the theorem.

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