GROWTH PROPERTY FOR THE MINIMAL SURFACE EQUATION IN UNBOUNDED DOMAINS

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Abstract. Here we prove that if \( u \) satisfies the minimal surface equation in an unbounded domain \( \Omega \) which is properly contained in a half plane, then the growth rate of \( u \) is of the same order as the shape of \( \Omega \) and \( u|_{\partial \Omega} \).

1. Introduction

The purpose of this paper is to improve a Phragmén-Lindelöf Theorem for the minimal surface equation in \( \mathbb{R}^2 \). Hwang has proved that if \( u \) satisfies the minimal surface equation in an unbounded domain \( \Omega \), which is properly contained in a half plane, the growth property of \( u \) depends on \( \Omega \) and \( u|_{\partial \Omega} \) only, without requiring any restriction for \( u \) [3]. In this respect, the Phragmén-Lindelöf Theorem for the minimal surface equation is better than that of the Laplace equation. We remark that if \( u \) satisfies the Laplace equation in an unbounded domain \( \Omega \), the growth property of \( u \) cannot be determined completely by the shape of \( \Omega \) and \( u|_{\partial \Omega} \) alone [7].

But the estimate in [3] is not good enough. For example, let \( \Omega = \{ - \cosh y < x < \cosh y | y > 0 \} \) and \( \text{div} \, Tu = 0 \) in \( \Omega \), \( u|_{\partial \Omega} = \sqrt{(\cosh y)^2 - x^2}|_{\partial \Omega} \). Then, by Example 3.4 of [3], we know that \( u = O(y e^y) \) as \( y \to \infty \), but the growth rate of the solution \( \sqrt{(\cosh y)^2 - x^2} \) (catenoid) is \( O(e^y) \).

The purpose of this paper is to improve the estimate of [3]. We will prove that the growth rate of \( u \) is of the same order as the shape of \( \Omega \) and \( u|_{\partial \Omega} \) (Theorems 2.12 and 2.13). In fact, let \( \Omega = \{ - \cosh y < x < \cosh y \} \). We will prove that a catenoid is the maximum solution among the solutions with vanishing boundary value (Corollary 2.3).

2. Phragmén-Lindelöf Theorems for \( \mathbb{R}^2 \)

Throughout the paper, \( \Omega \) will be a connected domain (bounded or unbounded) in \( \mathbb{R}^2 \) and, for any function \( u \in C^2(\Omega) \), \( Tu \) will denote the vector \( Du/\sqrt{1 + |Du|^2} \), where \( Du \) is the gradient vector of \( u \) and the minimal surface
operator $\mathcal{M}$ is given by

$$\mathcal{M}u = (1 + |Du|^2)\Delta u - D_i u D_j u D_i D_j u = (1 + |Du|^2)^{3/2} \operatorname{div} Tu,$$

where $\Delta u$ is the Laplacian of $u$.

We will use functions of the form

$$F(x, y) = (G(x, y))^{1/2} g(y) + h(y)$$

as comparison functions, and we compute $\mathcal{M} F$ in the following lemma.

**Lemma 2.1.** Let

$$F(x, y) = (G(x, y))^{1/2} g(y) + h(y),$$

where $F, G : \mathbb{R}^2 \to \mathbb{R}^1$, $g, h : \mathbb{R}^1 \to \mathbb{R}^1$, $F, G, g, h \in C^2$, and $G > 0$. Then $G^{3/2} \mathcal{M} F = I + II + III$, where

$$I = g^3 \left( \frac{1}{4} G_x^2 \left( \frac{1}{2} G_{yy} + G \frac{g''}{g} - 3G \left( \frac{g'}{g} \right)^2 \right) 
+ \frac{1}{2} G_{xx} \left( \frac{1}{4} G_y^2 + G_y \frac{g'}{g} + G^2 \left( \frac{g'}{g} \right)^2 \right) - \frac{1}{2} G_x G_{xy} \left( \frac{1}{2} G_y + G \frac{g'}{g} \right) \right),$$

$$II = g \left( -\frac{1}{4} G_y^2 + \frac{1}{2} G G_{yy} + G G_y \frac{g'}{g} + G^2 \frac{g''}{g} - \frac{1}{4} G_x^2 + \frac{1}{2} G_{xx} G \right),$$

$$III = G^{1/2} h'' \left( G + \frac{1}{4} G_x^2 g^2 \right) 
+ G^{1/2} h' \left( -\frac{1}{2} G_x G_{xy} g^2 - G_x^2 g g' + \frac{1}{2} G_y G_{xx} g^2 + G G_{xx} g g' \right) 
+ h' \left( -\frac{1}{4} G_x^2 g + \frac{1}{2} G G_{xx} g \right).$$

**Proof.** By simple computation we have

$$\left(1 + F_x^2\right) F_{yy} = \left(1 + \frac{1}{4} G^{-1} G_x^2 g^2 \right) \times \left( -\frac{1}{4} G^{-3/2} G_y g + \frac{1}{2} G^{-1/2} G_{yy} g + G^{-1/2} G_y g' + G^{1/2} g'' + h' \right),$$

$$-2 F_x F_y F_{xy} = -2 \left( \frac{1}{2} G^{-1/2} G_x g \right) \left( \frac{1}{2} G^{-1/2} G_y g + G^{1/2} g' + h' \right) \times \left( -\frac{1}{4} G^{-3/2} G_x G_y g + \frac{1}{2} G^{-1/2} G_{xy} g + \frac{1}{2} G^{-1/2} G_x g' \right),$$

$$(1 + F_y^2) F_{xx} = \left(1 + \left( \frac{1}{2} G^{-1/2} G_y g + G^{1/2} g' + h' \right)^2 \right) \times \left( -\frac{1}{4} G^{-3/2} G_x^2 g + \frac{1}{2} G^{-1/2} G_{xx} g \right).$$
Hence
\[
G^{5/2}MF = (G + \frac{1}{4}G_x^2g^2)(-\frac{1}{4}G_y^2g + \frac{1}{2}GG_yy_g + GG_yg' + G^2g'' + G^{3/2}h'') \\
- 2(\frac{1}{2}G_x g)(\frac{1}{2}G_y g + Gg')(-\frac{1}{4}G_x G_y g + \frac{1}{2}GG_{xy} g + \frac{1}{2}GG_x g') \\
+ (G + (\frac{1}{2}G_y g + Gg' + h'G^{1/2})^2)(-\frac{1}{4}G_x^2 g + \frac{1}{2}GG_{xx} g) \\
= \frac{1}{4}G_x^2 g^2(-\frac{1}{4}G_y^2 g + \frac{1}{2}GG_{yy} g + GG_yg' + G^2g'' \\
- 2(\frac{1}{2}G_x g)(\frac{1}{2}G_y g + Gg')(-\frac{1}{4}G_x G_y g + \frac{1}{2}GG_{xy} g + \frac{1}{2}GG_x g') \\
+ (\frac{1}{2}G_y g + Gg')^2(-\frac{1}{4}G_x^2 g + \frac{1}{2}GG_{xx} g) \\
+ G(-\frac{1}{4}G_y^2 g + \frac{1}{2}GG_{yy} g + GG_yg' + G^2g'' - \frac{1}{4}G_x^2 g + \frac{1}{2}GG_{xx} g)G \\
+ G^{3/2}h''(G + \frac{1}{4}G_x^2 g^2) \\
+ G^{1/2}h'(-G_x g)(-\frac{1}{4}G_x G_y g + \frac{1}{2}GG_{xy} g + \frac{1}{2}GG_x g') \\
+ 2h'G^{1/2}(\frac{1}{2}G_y g + Gg')(-\frac{1}{4}G_x^2 g + \frac{1}{2}GG_{xx} g) \\
+ \frac{1}{2}G(-\frac{1}{4}G_y^2 g + \frac{1}{2}GG_{xx} g) \\
= \frac{3}{4}G_x^2 g(\frac{1}{2}GG_{yy} g + G^2g'' - 3G^2(\frac{4}{g})^2) \\
+ \frac{1}{2}GG_{xx}(\frac{1}{2}G_y + Gg') - \frac{1}{2}GG_{xy}(\frac{1}{2}G_y + Gg') \\
+ G^2(-\frac{1}{4}G_x^2 g + \frac{1}{2}GG_{yy} g + GG_yg' + G^2g'' - \frac{1}{4}G_x^2 g + \frac{1}{2}GG_x G) \\
+ G(G^{1/2}h''(G + \frac{1}{4}G_x^2 g^2) \\
+ G^{1/2}h'(-\frac{1}{2}G_x G_{xy} g^2 - G^2g g' + \frac{1}{2}Gx G_{xx} g^2 + GG_{xx} g g')) \\
+ h^2(-\frac{1}{4}G_x^2 g + \frac{1}{2}GG_{xx} g) \\
= G(I + II + III).
\]

The lemma follows.

Now we treat the Phragmèn-Lindelöf Theorem for comparison functions $H(x, y)$ with faster growth. Since, in a half plane, the bound of the solutions with vanishing boundary value does not even exist, the domain must be properly contained in a half plane.

**Lemma 2.2.** Let $H(x, y) = a \sqrt{f^2(y)} - x^2$ and $\Omega \subset \{ -f_1(y) < x < f_1(y), y > 0 \}$, where $f, f_1 : [0, \infty) \to [0, \infty)$, $f \in C^2$, $f_1 \in C^0$, $f \geq f_1 > 0$ for $y > 0$, $f' \equiv \frac{df}{dy} > 0$, and $a$ is a positive constant. Let $u \in C^2(\Omega) \cap C^0(\Omega)$, and suppose that

(i) $\text{div} \, TH - \text{div} \, Tu \leq 0$ in $\Omega$,

(ii) $(u - H)_{|\partial \Omega} \leq 0$,

(iii) $\liminf_{y \to \infty} f_1(y) / f'^2 = 0$.

Then $u \leq H$ in $\Omega$.

**Remark.** It is easy to see that if $\lim_{y \to \infty} f(y)/f'^2 = 0$, then the rate of growth of $f$ must be faster than $y^2$ as $y \to \infty$.

**Proof of Lemma 2.2.** If $\{ (x, y) \in \Omega | u(x, y) - H(x, y) > 0 \}$ is nonempty, there exists $\varepsilon > 0$ such that $\Omega' = \{ (x, y) \in \Omega | u(x, y) - H(x, y) > \varepsilon \}$ is non-empty and $\partial \Omega' \cap \Omega$ is smooth (Sard's Theorem). Since $(u - H)_{|\partial \Omega} \leq 0$, we have $\partial \Omega' \subset \Omega$ and $\partial \Omega' = \{ (x, y) \in \Omega | u - H = \varepsilon \}$.
For every $y_0 > 0$, let $\Omega_{y_0} = \Omega' \cap \{ y < y_0 \}$ and $\Gamma_{y_0} = \partial \Omega_{y_0} \cap \{ y = y_0 \}$. By the divergence theorem, we have

$$\int_{\partial \Omega_{y_0}} \tan^{-1}(u - H - \varepsilon)(Tu - TH) \cdot \nu d\sigma$$

$$= \int \int_{\Omega_{y_0}} \frac{(Du - DH)}{1 + (u - H - \varepsilon)^2} (Tu - TH) dx$$

$$+ \int \int_{\Omega_{y_0}} (u - H - \varepsilon)(\text{div } Tu - \text{div } TH) dx,$$

where $\nu$ is the unit outer normal of $\partial \Omega_{y_0}$. Noticing that $\partial \Omega_{y_0} \setminus \Gamma_{y_0} \subset \partial \Omega'$, we have $u - H = \varepsilon$ on $\partial \Omega_{y_0} \setminus \Gamma_{y_0}$. Since $\tan^{-1}(u - H - \varepsilon)(\text{div } Tu - \text{div } TH) \geq 0$ in $\Omega_{y_0}$ and $Tu \cdot \nu \leq 1$,

$$\int \int_{\Omega_{y_0}} \frac{(Du - DH)}{1 + (u - H - \varepsilon)^2} (Tu - TH) dx$$

$$\leq \int_{\Gamma_{y_0}} \tan^{-1}(u - H - \varepsilon)(Tu - TH) \cdot \nu d\sigma$$

$$\leq \int_{\Gamma_{y_0}} \tan^{-1}(u - H - \varepsilon)(1 - TH \cdot \nu) d\sigma.$$

Since $\nu = (0, 1)$ on $\Gamma_{y_0}$,

$$H = a\sqrt{f^2 - x^2}, \quad H_y = a\frac{f f'}{\sqrt{f^2 - x^2}}, \quad H_x = -\frac{a x}{\sqrt{f^2 - x^2}},$$

$$TH \cdot \nu|_{\Gamma_{y_0}} = \frac{H_y}{\sqrt{1 + H_x^2 + H_y^2}} = \frac{a f f' / \sqrt{f^2 - x^2}}{\sqrt{1 + a^2 x^2 / (f^2 - x^2) + a^2 f^2 f'^2 / (f^2 - x^2)}}$$

$$= \frac{a f f'}{\sqrt{f^2 - x^2 + a^2 x^2 + a^2 f^2 f'^2}},$$

$$(1 - TH \cdot \nu)|_{\Gamma_{y_0}}$$

$$= \frac{\sqrt{f^2 - x^2 + a^2 x^2 + a^2 f^2 f'^2 - a f f'}}{\sqrt{f^2 - x^2 + a^2 x^2 + a^2 f^2 f'^2} \cdot \sqrt{f^2 - x^2 + a^2 x^2 + a^2 f^2 f'^2 + a f f'}}$$

$$= \frac{f^2 - x^2 + a^2 x^2}{\sqrt{f^2 - x^2 + a^2 x^2 + a^2 f^2 f'^2}}$$

$$\leq \frac{(1 + a^2) f^2}{a^2 f^2 f'^2} = \frac{1 + a^2}{a^2 f^2 f'^2}.$$
Now
\[ \int_{\Omega_0} \frac{(Du - DH)}{1 + (u - H - \varepsilon)^2} (Tu - TH) \, dx \]
\leq \int_{\Gamma_{\Omega_0}} \tan^{-1}(u - H - \varepsilon)(1 - TH \cdot \nu) \, d\sigma
\leq \frac{\pi}{2} \int_{\Gamma_{\Omega_0}} \frac{1 + a^2}{a^2 f'^2} \, d\sigma \leq \pi \cdot \frac{(1 + a^2)}{a^2 f'^2}(y_0).

Let \( y_0 \to \infty \). We have
\[ 0 \leq \int_{\Omega} \frac{(Du - DH)}{1 + (u - H - \varepsilon)^2} (Tu - TH) \, dx \leq 0. \]

Since \( (Du - DH) \cdot (Tu - TH) \geq 0 \) and the equality holds when \( Du = DH \), we have \( Du - DH \equiv 0 \) in \( \Omega' \). Then \( u \equiv H + \varepsilon \) in \( \Omega' \), and, by definition, \( \Omega' \) must be empty. This is impossible, and we conclude that \( u(x) \leq H(x) \) for all \( x \) in \( \Omega \).

Remark. The following well-known fact is used to prove Lemma 2.2: \( Du/\sqrt{1 + |Du|^2} \) has norm less than 1. It is a very important idea for the capillary surface equation (cf. [1, Theorem 5.1]).

Now we obtain the result: catenoid is the maximum solution among those surfaces on \( \Omega = \{- \cosh y < x < \cosh y\} \) satisfying the minimal surface equation and with vanishing boundary value.

**Corollary 2.3.** Let \( \Omega = \{- \cosh y < x < \cosh y\} \), and let \( u \in C^2(\Omega) \cap C^0(\Omega) \).

Suppose that
(i) \( \text{div} \, Tu > 0 \) in \( \Omega \),
(ii) \( u|_{\partial \Omega} \leq 0 \).

Then \( u \leq \sqrt{(\cosh y)^2 - x^2} \).

**Proof.** The corollary can be proved by the fact that
\[ \text{div} \, Tu - \text{div} \, T\sqrt{(\cosh y)^2 - x^2} \geq 0, \quad \lim_{y \to \pm \infty} \frac{\cosh y}{[(\cosh y)^2]^2} = 0, \]
and a similar argument as in the proof of Lemma 2.2.
(ii) If \( p_0 < 0 \), \( a > \sqrt{1 - p_0} \), \( a \) is a positive constant, and \( f^2 \geq (a^2 - 1)(2 - p_0)f_1^2/(a^2 - (1 - p_0)) \), we have \( G^{3/2} \mathcal{M} F \leq 0 \) in \( \Omega \).

Proof. (i) If \( 2 \geq p(f) \geq p_0 \), it is easy to see that

\[
G^{3/2} \mathcal{M} F \leq 0.
\]

If \( p(f) \geq 2 \), we have

\[
G^{3/2} \mathcal{M} F \leq (p(f) - 2)\sqrt{1 - p_0}p_0 x^2 f^2 + f^2 f_1^2 \sqrt{1 - p_0}(-p(f) + p_0) \leq 0.
\]

(ii) By assumption, it is easy to have the following:

\[
G^{3/2} \mathcal{M} F = f^2 a^3 (-f^2 + (2 - p(f))x^2)
\]

\[
+ f^2 a(f^2(1 - p(f)) - (2 - p(f))x^2) - af^2
\]

\[
= af^2 ((a^2 - 1)(2 - p(f))x^2 - f^2(a^2 - (1 - p(f)))) - af^2 \leq 0.
\]

Hence it is easy to derive the following theorem.

**Theorem 2.5.** Let \( f, f_1, p(f), \) and \( \Omega \) be defined as in Lemma 2.4. Let \( \liminf_{\gamma \to \infty} (f_1/f^2) = 0, u \in C^2(\Omega) \cap C^0(\Omega), \) and \( \text{div} \, Tu \geq 0 \) in \( \Omega \). Then

(i) If \( p(f) \geq p_0 \geq 0 \) where \( p_0 \) is a constant, \( f \geq f_1, 1 \geq p_0 \geq 0, \) and \( u|_{\partial \Omega} \leq \sqrt{1 - p_0} \sqrt{f^2 - x^2}|_{\partial \Omega} \), then we have \( u \leq \sqrt{1 - p_0} \sqrt{f^2 - x^2} \) in \( \Omega \).

(ii) If \( p(f) \geq p_0 \), where \( p_0 \) is a negative constant, and

\[
f^2 \geq (a^2 - 1)(2 - p_0)f_1^2, \quad u|_{\partial \Omega} \leq a \sqrt{f^2 - x^2}|_{\partial \Omega},
\]

where \( a \) is a positive constant satisfying \( a^2 - 1 + p_0 > 0 \), then we have \( u \leq a \sqrt{f^2 - x^2} \) in \( \Omega \).

We will show later that the above theorem still remains valid without the condition \( \liminf_{\gamma \to \infty} (f_1/f^2) = 0 \). We first investigate the properties of \( p(f) \). Let \( f \in C^2, f \geq 0, f' \geq 0, a \) be a positive constant, and \( \alpha \neq 0 \) be a constant. Since \( p(af^\alpha) = (1/(\log(af^\alpha))')' = \frac{1}{\alpha}(1/(\log f')')' \), we have \( p(af^\alpha) = \frac{1}{\alpha} p(f) \). Then we prove the following lemma.

**Lemma 2.6.** Let \( f, f_1 \in C^2 \), and \( f, f', f_1, f_1' > 0 \). Then

(i) If \( p(f), p(f_1) \geq p_0 > 0 \), where \( 0 < p_0 \leq 1 \), is a constant, then \( p(ff_1) \geq p_0/2 > 0 \);

(ii) If \( p(f) \geq 0 \), then \( p(e^{ay}f) \geq 0 \) for any positive constant \( a \);\n
(iii) If \( p(f) \geq p_0 \), where \( p_0 \) is a negative constant, then \( p(e^{ay}f) \geq p_0 \) for any positive constant \( a \).

Proof. (i) Let \( h = (\log f)' = f'/f > 0 \) and let \( h_1 = (\log f_1)' = f_1'/f_1 > 0 \). Since \( p(f) = (1/h) = -h'/h^2 \geq p_0 > 0 \) and \( p(f_1) = (1/h_1)' = -h_1'/h_1^2 \geq p_0 > 0 \), we have \( p(ff_1) = (1/(h + h_1))' = (h' + h_1')/(h + h_1)^2 \geq (h' + h_1')/(2(h^2 + h_1^2)) \geq p_0/2 \).

(ii) Since \( p(f) = -h'/h^2 \geq 0 \), we have \( p(e^{ay}f) = (1/((\log e^{ay})' + (\log f)'))' = (1/(a + h))' = -h/(a + h)^2 \geq 0 \).

(iii) Since \( p(f) = -h'/h^2 \) and \( p(e^{ay}f) = -h/(a + h)^2 \) have the same sign and \( |p(e^{ay}f)| \leq |p(f)| \), the result follows immediately.
Remark 2.7. (i) If \( f = y^m \), where \( m \) is a positive constant, then \( p(f) = \frac{1}{m} \).

(ii) \( p(e^y) = 0 \).

(iii) If we set \( f = e^{\alpha y} \), where \( \alpha > 1 \) is a constant, then \( p(f) = (1/(y^\alpha))' = (y^{1-\alpha}/\alpha)' = (1-\alpha)\alpha^{-\alpha}/\alpha \). This implies that \( p(f) \to 0^- \) as \( y \to \infty \), so, for sufficiently large \( y \), we have \( p(f) > -\varepsilon \) for some small positive number \( \varepsilon \).

Similarly, if \( f \) increases faster than the exponential function, we can assume \( p(f) \geq -\varepsilon \) for some small positive constant \( \varepsilon \) essentially.

Lemma 2.8. Let \( f \in C^1 \), \( f' > 0 \), and \( \limsup_{y \to \infty} (f(y)/y^2) = +\infty \). Then \( \liminf_{y \to \infty} (f/f^2) = 0 \).

Proof. Suppose not; then there exist positive constants \( y_0 \) and \( C \) such that, for every \( y > y_0 \), we have \( f(y)/f^2(y) \geq C \), so \( f = O(y^2) \). Contradiction arises and the lemma follows.

We are now ready to remove the condition \( \liminf_{y \to \infty} (f/f^2) = 0 \) in Theorem 2.5.

We will start with a theorem.

Theorem 2.9. Let \( f \in C^0[0, \infty) \cap C^2(0, \infty) \), \( f \geq 0 \), \( f' > 0 \), \( \Omega \subset \{(x, y)\mid -f(y) < x < f(y), y > 0\} \), \( u \in C^2(\Omega) \cap C^0(\Omega) \), \( \text{div} \, Tu \geq 0 \), and \( p(f) \geq p_0 > 0 \), where \( p_0 \) is a constant. Then

(i) if \( p_0 = 0 \) and \( (u - y/P - x^2)|_{\partial \Omega} < 0 \), we have \( u < \sqrt{h^2 - x^2} \) in \( \Omega \).

(ii) if \( 1 \geq p_0 > 0 \) and \( (u - \sqrt{1 - q_0\sqrt{h^2 - x^2}})|_{\partial \Omega} < 0 \), where \( q_0 = \min(1/4, p_0/2) \), we have \( u < \sqrt{1 - q_0\sqrt{h^2 - x^2}} \) in \( \Omega \).

Proof. (i) Let \( h = e^{ay}f \), where \( a \) is a positive constant. Then by Lemma 2.6 \( p(h) \geq 0 \). So \( \mathfrak{m}(\sqrt{h^2 - x^2}) \leq 0 \) in \( \Omega \). Since \( f' > 0 \), \( \limsup_{y \to \infty} (f/y^2) = +\infty \). By Lemma 2.8, \( \liminf_{y \to \infty} (h/h^2) = 0 \). Since \( u|_{\partial \Omega} \leq \sqrt{h^2 - x^2}|_{\partial \Omega} \leq \sqrt{h^2 - x^2}|_{\partial \Omega} \), by Theorem 2.5, \( u \leq \sqrt{h^2 - x^2} = \sqrt{(e^{ay}f)^2 - x^2} \) in \( \Omega \). Setting \( a \to 0 \), the result follows.

(ii) Let \( h = (1 + by)^4f \), where \( b \) is a positive constant. Since \( p((1 + by)^4) = \frac{1}{4} \), by Lemma 2.6, \( p(h) \geq \min(1/4, p_0)/2 = q_0 > 0 \). By Lemma 2.4, \( \mathfrak{m}(\sqrt{1 - q_0\sqrt{h^2 - x^2}}) \leq 0 \) in \( \Omega \). Since \( f' > 0 \), \( \limsup_{y \to \infty} (h/y^2) = +\infty \). So \( \liminf_{y \to \infty} (h/h^2) = 0 \). Since

\[
\frac{d}{d \Omega} \leq \sqrt{1 - q_0\sqrt{h^2 - x^2}}|_{\partial \Omega} \leq \sqrt{1 - q_0\sqrt{h^2 - x^2}}|_{\partial \Omega},
\]

by Theorem 2.5, \( u \leq \sqrt{1 - q_0\sqrt{h^2 - x^2}} = \sqrt{1 - q_0\sqrt{(1 + by)^8f^2 - x^2}} \) in \( \Omega \); the result then follows by letting \( b \to 0 \).

Theorem 2.10. Let \( f, \Omega, \) and \( u \) be as in Theorem 2.9. \( \lim_{y \to \infty} f(y) = +\infty \), and \( p(f) \geq p_0 > 0 \), where \( 1 \geq p_0 > 0 \) is a constant. Suppose \( u|_{\partial \Omega} \leq \sqrt{1 - p_0\sqrt{f^2 - x^2}} \). Then \( u \leq \sqrt{1 - p_0\sqrt{f^2 - x^2}} \) in \( \Omega \).

Proof. Case 1: \( 0 < p_0 < 1 \). Let \( q_0 = \min(1/8, p_0/2) \). Then 

\[
\frac{d}{d \Omega} \leq \sqrt{1 - p_0\sqrt{f^2 - x^2}}|_{\partial \Omega} \leq \sqrt{1 - q_0\sqrt{f^2 - x^2}}|_{\partial \Omega},
\]
By Theorem 2.9, \( u \leq \sqrt{1 - q_0 \sqrt{f^2 - x^2}} \) in \( \Omega \). Let
\[
F = \sqrt{f^2 - x^2 - bh(y)^{-\alpha}},
\]
where \( \alpha \) is some constant in \((0, 1)\) to be determined later, \( \varepsilon \) is a constant such that \( 0 < \varepsilon < p_0 \) and \( b = p_0 - \varepsilon \), and \( h(y) = f(y_0 + y) \), where \( y_0 \) is a positive constant. Since \( f'' > 0 \) and \( \lim_{y \to \infty} f = +\infty \), we can choose \( y_0 > 0 \) such that \( h \geq 1 \) for every \( y > 0 \). Let
\[
G = f^2 - x^2, \quad g = (1 - bh^{-\alpha})^{1/2},
\]
\[
g' = \frac{1}{2}(1 - bh^{-\alpha})^{-1/2}(-b)(-\alpha)h^{-\alpha-1}h',
\]
\[
g'' = -\frac{1}{4}(1 - bh^{-\alpha})^{-3/2}(-b)(-\alpha)h^{-\alpha-1}h' + b(1 - bh^{-\alpha})^{-1/2}((-\alpha - 1)h^{-\alpha-2}h'^2 + h^{-\alpha-1}h'').
\]
But \( p(h) \geq p_0 > 0 \), so \( h'^2 \geq hh'' \). Hence \( g'' \leq 0 \). By Lemma 2.1,
\[
G^{3/2} \mathcal{M} F = g \left( -f^2 f'' + (f^2 - x^2)(f^2 + f f'') - x^2 - (f^2 - x^2) + (f^2 - x^2)2 f f' \frac{g'}{g} + G^2 \frac{g''}{g} \right)
\]
\[
\leq g(-f^2 f'' + (f^2 - x^2)(f^2 + f f'') - x^2 - (f^2 - x^2) + 2 f f' \frac{g'}{g} (f^2 - x^2))
\]
\[
+ g^3(x^2(f^2 + f f'') - f^2 f'^2)
\]
\[
\leq g + g^3(2 - p(f))x^2 f^2 + f^2 f^2 g((1 - p(f)) - g^2)
\]
\[
- g f^2 + 2 f f' g' (f^2 - x^2)
\]
\[
\leq -g + g^3(2 - p(f))x^2 f^2 - f^2 f^2 g \max(p(f) - 2, 0)
\]
\[
+ f^2 f^2 g((1 - \min(2, p(f)) - g^2) - g f^2 + 2 f f' g' (f^2 - x^2)).
\]
Since \( 1 \geq g \geq \sqrt{1 - b} \),
\[
G^{3/2} \mathcal{M} F \leq f^2 f^2 \frac{g}{g^2} \frac{x^2}{g^2} + 2 f f' \frac{g'}{g} f^2
\]
\[
\leq f^2 f^2 g(-\min(2, p(f)) + bh^{-\alpha}) + 2 f^3 f' \frac{1}{2}(1 - bh^{-\alpha})^{-1/2} b\alpha h^{-\alpha-1}h'
\]
\[
\leq f^2 f^2 \sqrt{1 - b}(-\min(2, p(f)) + bh^{-\alpha}) + b\alpha h^{-\alpha-1}h' \frac{1}{\sqrt{1 - b}} f^3 f'
\]
\[
\leq f^2 f^2 \sqrt{1 - b}(-\min(2, p(f)) + b)h^{-\alpha} + b\alpha h^{-\alpha-1}h' \frac{1}{\sqrt{1 - b}} f^3 f'
\]
\[
\leq h^{-\alpha} f^2 f^2 \left(-\sqrt{1 - b} + \alpha bh^{-1}h' \frac{1}{\sqrt{1 - b}} f' \right).
\]
Since \((f/f')' = p(f) > 0\), \(f/f'\) is monotone increasing. But \(h(y) = f(y_0 + y), y_0\) is a positive constant; therefore \(h^\prime f \leq 1\). So when \(0 < \alpha < (1-b)e/b\), we have \(G^{3/2} \mathcal{M} \leq 0\). Since

\[
\lim_{y \to \infty} f(y) = +\infty, \quad \lim_{y \to \infty} \sqrt{1 - bh(y)^{-\alpha}} \to 1,
\]

we have \(F(x, y) \geq \sqrt{1 - q_0 \sqrt{f^2 - x^2}}\) for sufficiently large \(y\). Since \(u \leq \sqrt{1 - q_0 \sqrt{f^2 - x^2}}\) in \(\Omega\) and \(u|_{\partial \Omega} \leq F|_{\partial \Omega}\) by hypothesis, we have \(u \leq F\) in \(\Omega\). So \(u \leq \sqrt{1 - bh^{-\alpha} \sqrt{f^2 - x^2}}\) in \(\Omega\). Letting \(\alpha \to 0\), \(u \leq \sqrt{1 - b \sqrt{f^2 - x^2} = \sqrt{1 - (p_0 - e) \sqrt{f^2 - x^2}}\) in \(\Omega\). Letting \(e \to 0\), we get \(u \leq \sqrt{1 - p_0 \sqrt{f^2 - x^2}}\) in \(\Omega\).

**Case 2:** \(p_0 = 1\). Then for any constant \(p_1, 0 < p_1 < 1\), we have \(p(f) \geq 1 \geq p_1\). Since \(u|_{\partial \Omega} \leq \sqrt{1 - 1 \sqrt{f^2 - x^2} \leq \sqrt{1 - p_1 \sqrt{f^2 - x^2}}\), by Case 1, we have \(u \leq \sqrt{1 - p_1 \sqrt{f^2 - x^2}}\) in \(\Omega\). Letting \(p_1 \to 1\), the result follows.

Since the case \(\lim f < +\infty\) is not very important, we omit that case.

The case for a negative constant \(p_0\) is studied in the following theorem.

**Theorem 2.11.** Let \(f \in C^0[0, \infty) \cap C^2(0, \infty), f \geq 0, f'' > 0, f_1 \in C^0[0, \infty), f_1 > 0, \Omega \subset \{(x, y)| - f_1(y) < x < f_1(y), y > 0\}, u \in C^2(\Omega) \cap C^0(\Omega), \text{div} \nabla u \geq 0\) in \(\Omega,\) and \(p(f) \geq p_0,\) where \(p_0\) is a negative constant. Then if \(f^2 \geq (a^2 - 1)(2 - p_0)f_1/(a^2 - (1 - p_0))\) and \(u|_{\partial \Omega} \leq a \sqrt{f^2 - x^2}|_{\partial \Omega},\) where \(a\) is a positive constant satisfying \(a^2 - 1 + p_0 > 0,\) we have \(u \leq a \sqrt{f^2 - x^2}\) in \(\Omega.\)

**Proof.** Let \(f_2 = e^{a y} f,\) where \(a\) is a positive constant. Then \(f_2^2 \geq f^2 \geq (a^2 - 1)(2 - p_0)f_1/(a^2 - (1 - p_0)).\) By Lemma 2.6, \(p(f_2) \geq p_0.\) It is easy to see that \(\limsup_{y \to \infty} (f_2/y)^2 = +\infty,\) by Lemma 2.8, and \(\liminf_{y \to \infty} (f_2/y)^2 = 0.\)

Since \(u|_{\partial \Omega} \leq a \sqrt{f^2 - x^2}|_{\partial \Omega} \leq a \sqrt{f_1^2 - x^2}|_{\partial \Omega},\) by Theorem 2.5, \(u \leq a \sqrt{f^2 - x^2}\) = \(a \sqrt{e^{a y} f^2 - x^2}\) in \(\Omega.\) The result then follows by letting \(a \to 0.\)

Let \(\Omega \subset \{(x, y)| - f_1(y) < x < f_1(y), y > 0\}\) and \(u|_{\partial \Omega} \leq a f_1,\) where \(a\) is a positive constant. The growing rate of \(u\) is stated in the following theorem.

**Theorem 2.12.** Let \(f_1 \in C^0[0, \infty) \cap C^2(0, \infty), f_1, f_1^\prime > 0, \lim_{y \to \infty} f_1 = \infty,\) and \(p(f_1) \geq p_0 \geq 0,\) where \(1 \geq p_0 \geq 0\) is a constant. Moreover, let \(\Omega \subset \{(x, y)| - f_1(y) < x < f_1(y), y > 0\}\) and \(f = a f_1,\) where \(a \geq 0\) is a constant. Then if \(\text{div} \nabla u \geq 0\) in \(\Omega\) and \(u|_{\partial \Omega} \leq f,\) we have \(u \leq \sqrt{(a^2 + 1 - p_0)f_1^2 + (1 - p_0)x^2}\) in \(\Omega.\)

**Proof.** Case 1: \(1 > p_0 \geq 0.\) Since

\[
u|_{\partial \Omega} \leq f = a f_1 \leq \sqrt{1 - p_0} \sqrt{\frac{a^2 f_1^2}{1 - p_0} + f_1^2 - x^2}|_{\partial \Omega}
\]

and \(p(\sqrt{\frac{a^2}{1 - p_0} + f_1}) = p(f) \geq p_0,\)

\[
u \leq \sqrt{1 - p_0} \sqrt{\frac{a^2}{1 - p_0} f_1^2 + f_1^2 - x^2} = \sqrt{(a^2 + (1 - p_0))f_1^2 - (1 - p_0)x^2}\] in \(\Omega\)

by Theorem 2.10.
Case 2: \( p_0 = 1 \). For every constant \( b \) where \( 1 \geq b \geq 0 \), by Case 1, we have \( u \leq \sqrt{(a^2 + (1 - b))f_i^2 - (1 - b)x_i^2} \) in \( \Omega \). The result then follows by letting \( b \to 0 \).

Now we assume that \( p(f_i) \geq p_0 \) with \( p_0 \) being a negative constant, and set \( f^2 = (a^2 - 1)(2 - p_0)/f_i^2 / (a^2 - (1 - p_0)) \), where \( a > \sqrt{1 - p_0} \) is a constant. If \( u|_{\partial \Omega} \leq a \sqrt{f^2 - x_i^2}|_{\partial \Omega} \leq a \sqrt{(a^2 - 1)(2 - p_0)/(a^2 - (1 - p_0)) - 1} \), by Theorem 2.11, we have \( u \leq a \sqrt{f^2 - x_i^2} \) in \( \Omega \). Now we want to compute the minimum of \( a \sqrt{(a^2 - 1)(2 - p_0)/(a^2 - (1 - p_0)) - 1} \). For convenience, let \( b = a^2 \) and \( q_1 = 1 - p_0 \), and we need to compute the minimum of

\[
\frac{(b - 1)(1 + q_1) - 1}{b - q_1}.
\]

Take logarithms and differentiate the above function; we have \( 1/b + q_1/(q_1 b - 1) = 1/(b - q_1) \). From this, we get \( b = q_1 \pm \sqrt{q_1^2 - 1} \). Since \( b = a^2 > q_1 \), we have \( b = q_1 + \sqrt{q_1^2 - 1} \). So the minimum of \( b(q_1 b - 1)/(b - q_1) \) is

\[
(q_1 + \sqrt{q_1^2 - 1}) \left( \frac{q_1^2 + q_1 \sqrt{q_1^2 - 1} - 1}{\sqrt{q_1^2 - 1}} \right) = (q_1 + \sqrt{q_1^2 - 1})^2,
\]

and the minimum of \( a \sqrt{(a^2 - 1)(2 - p_0)/(a^2 - (1 - p_0)) - 1} \) is \( q_1 + \sqrt{q_1^2 - 1} \).

**Theorem 2.13.** Assume that \( f_i, f_i' > 0, p(f_i) \geq p_0 \), where \( p_0 \) is a negative constant, \( \Omega \subset \{ -f_i(y) < x < f_i(y)|y > 0 \} \), and \( u \in C^2(\Omega) \cap C^0(\Omega) \). If \( \text{div} \, Tu \geq 0 \) in \( \Omega \), \( q_1 = 1 - p_0 \geq 1 \), and \( u|_{\partial \Omega} \leq b f_i \), where \( b \geq q_1 + \sqrt{q_1^2 - 1} \) is a positive constant, then \( u \leq \sqrt{b} \sqrt{(b + 1)f^2 - x_i^2} \) in \( \Omega \).

**Proof.** By the assumption that \( b \geq q_1 + \sqrt{q_1^2 - 1} \), we have

\[
(b - (q_1 + \sqrt{q_1^2 - 1}))(b - (q_1 - \sqrt{q_1^2 - 1})) \geq 0,
\]

\[b^2 - 2q_1 b + 1 \geq 0, \text{ and } (b + 1)(b - q_1) \geq (b - 1)(1 + q_1). \] If we set \( a = b^2 \), then \( a^2 + 1 \geq (a^2 - q_1) \geq (a^2 - 1)(1 + q_1) \) and we have \( a^2 + 1 \geq (a^2 - 1)(1 + q_1)/(a^2 - q_1) \). Since \( u|_{\partial \Omega} \leq a \sqrt{(a^2 + 1)f_i^2 - x_i^2}|_{\partial \Omega} \), by Theorem 2.11, \( u \leq a \sqrt{(a^2 + 1)f_i^2 - x_i^2} \) in \( \Omega \) and we have \( u \leq \sqrt{b} \sqrt{(b + 1)f^2 - x_i^2} \) in \( \Omega \).

By Theorems 2.12 and 2.13, we know that the growth rate of \( u \) is of the same order as the shape of \( \Omega \) and \( u|_{\partial \Omega} \).

**Acknowledgment**

The author would like to thank the referee for many helpful comments and suggestions.
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