LINKED PAIRS OF CONTRACTIBLE POLYHEDRA IN $S^n$

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(Communicated by James West)

Abstract. B. Mazur has described a geometrically linked pair of compact contractible polyhedra in $S^4$. In this note we exhibit an even more extreme type of linking between compact contractible polyhedra in $S^n$, $n \geq 5$.

1. Introduction

Disjoint compacta $A_1, A_2 \subset S^n$ are geometrically unlinked if there is a PL embedding $f: S^{n-1} \to S^n$ so that $f(S^{n-1})$ separates $S^n$ into components $V_1$ and $V_2$ with $A_1 \subset V_1$ and $A_2 \subset V_2$. In this case, $V_1$ and $V_2$ are contractible polyhedra (see (2) from §2), so by taking interiors of sufficiently small regular neighborhoods of $V_1$ and $V_2$ we see that if $A_1$ and $A_2$ are geometrically unlinked they also satisfy

Definition. Disjoint compacta $A_1, A_2 \subset S^n$ are fundamentally unlinked if there is a cover $\{U_1, U_2\}$ of $S^n$ by contractible open sets so that $A_i \subset U_i$ for $i = 1, 2$ and $A_i \cap U_j = \emptyset$ when $i \neq j$.

If $A_1$ and $A_2$ are disjoint compact contractible polyhedra in $S^n$ and $n \leq 3$, then they are geometrically unlinked. Indeed, if $N(A_1)$ is a regular neighborhood of $A_1$ disjoint from $A_2$, then $\partial N(A_1)$ is a PL $(n-1)$-sphere separating $A_1$ from $A_2$. In [Ma] Mazur made the surprising observation that, in $S^4$, a disjoint pair of compact contractible polyhedra may be geometrically linked. To do this, he constructed a compact contractible 4-manifold $M$ (now known as a “Mazur manifold”) which has nonsimply connected boundary and may be viewed as a regular neighborhood of a contractible 2-complex $D$ contained in its interior. He then observes that the double, $M_1 \cup_\partial M_2$, of $M$ is a PL 4-sphere and $D_1$ and $D_2$ are geometrically linked therein. Notice, however, that $D_1$ and $D_2$ are fundamentally unlinked.

A strategy similar to Mazur’s may be used to produce pairs of geometrically linked, but fundamentally unlinked, compact contractible polyhedra in $S^n$ for all $n \geq 4$. In this note we show that for $n \geq 5$ there exist fundamentally linked pairs of compact contractible polyhedra in $S^n$.  

Received by the editors November 6, 1992.

1991 Mathematics Subject Classification. Primary 57N15, 57Q99.

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0002-9939/94 $1.00 + .25$ per page
2. Preliminaries

Throughout this paper we work in the PL category; all complexes are simplicial, manifolds are combinatorial, and maps are piecewise linear. All homology is with \( \mathbb{Z} \)-coefficients.

A group \( G \) is perfect if its abelianization, \( G/[G, G] \), is the trivial group. A space \( X \) is acyclic if \( \tilde{H}_k(X) = 0 \) for all \( k \). A compact acyclic \( n \)-manifold is called a homology \( n \)-cell. An \( n \)-manifold with homology groups isomorphic to those of \( S^n \) is called a homology \( n \)-sphere.

The following facts are well known. They follow from standard results of algebraic topology including the VanKampen, Mayer-Vietoris, and Universal Coefficient theorems, as well as duality, the Hurewicz Theorem, and a theorem of Whitehead. We list them here for easy reference.

(1) The boundary of a homology \( n \)-cell is a homology \( (n-1) \)-sphere.

(2) If \( \Sigma^{n-1} \subset S^n \) is a homology \( (n-1) \)-sphere and \( V_1 \) and \( V_2 \) are the components of \( S^n - \Sigma^{n-1} \), then \( V_1 \) and \( V_2 \) are acyclic. If \( \Sigma^{n-1} \) is simply connected, then \( V_1 \) and \( V_2 \) are simply connected and thus contractible. If \( \Sigma^{n-1} \) is locally flat, then \( V_1 \) and \( V_2 \) are homology \( n \)-cells.

(3) The union of two homology \( n \)-cells among a common boundary is a homology \( n \)-sphere.

3. Main result

Theorem 3.1. For any \( n \geq 5 \), there exists a fundamentally linked pair of compact contractible polyhedra in \( S^n \).

We will need the following lemmas. Both are tailored to the proof of Theorem 3.1 and could be stated in greater generality if so desired.

Lemma 3.2. Let \( K \) be a finite acyclic 2-complex with fundamental group \( G \). Then, for any \( n \geq 5 \), there exists a homology \( n \)-sphere \( \Sigma^n \) with \( \pi_1(\Sigma^n) \cong G \times G \).

Proof. For \( n \geq 8 \), we may embed \( K \times K \) in \( \mathbb{R}^{n+1} \). A regular neighborhood \( N \) of this embedding is a homology \((n+1)\)-cell, so, by (1), \( \partial N \) is a homology \( n \)-sphere; moreover, by general position, \( \pi_1(\partial N) \cong \pi_1(N) \cong G \times G \). Now, since \( G \times G \) is the fundamental group of some high-dimensional homology sphere, the proof of Theorem 1 in [Ke], together with the remarks that precede it, show implicitly that there is an acyclic 3-complex, \( L \), with \( \pi_1(L) \cong G \times G \). Hence, for \( n \geq 6 \), we may use the same strategy as above. Finally, for \( n = 5 \), apply [St] to obtain a 3-complex \( L' \subset \mathbb{R}^6 \) which is simple homotopy equivalent to \( L \), and let \( \Sigma^n \) be the boundary of a regular neighborhood of \( L' \). \( \square \)

Remark. Nonsimply connected, acyclic 2-complexes are plentiful. For example, removing the interior of a 3-ball from a nonsimply connected homology 3-sphere produces a homology 3-cell with the same fundamental group. This homology cell may then be collapsed onto a 2-dimensional subcomplex.

Lemma 3.3. Let \( K \) be a finite complex with perfect fundamental group \( G \). If \( K \) may be written as \( U \cup V \), where \( U \) and \( V \) are open (not necessarily connected) subsets of \( K \), such that loops lying completely within either \( U \) or \( V \) contract in \( K \), then \( K \) is simply connected.

Proof. By [Wr, Lemma 7.2], \( G \) must be a free group, but the only perfect free group is trivial. \( \square \)
Proof of Theorem 3.1. Let $K$ be an acyclic 2-complex with nontrivial fundamental group $G$. By Lemma 3.2, we may choose a homology $n$-sphere, $\Sigma^n$ with $\pi_1(\Sigma^n, q) \cong G \times G$. Let $G_1, G_2, G_3 < \pi_1(\Sigma^n, q)$ correspond to $G \times \{1\}$, $\{1\} \times G$, $\Delta_G = \{(g, g) | g \in G\} < G \times G$, respectively. Choose PL embeddings $e_i : (K, p) \to (\Sigma^n, q)$ for $i = 1, 2, 3$ so that image($e_i$) = $\pi_1(K, p) \to \pi_1(\Sigma^n, q)$ = $G_i$ for each $i$. By general position, we may homotope $e_1$ and $e_2$ to embeddings $e'_1$ and $e'_2$ so that $e'_1$ and $e'_2$, and $e_3$ have pairwise disjoint images which we will denote by $K_1, K_2$, and $K_3$. Choose regular neighborhoods $N_1$ and $N_2$ of $K_1$ and $K_2$ so that $N_1$, $N_2$, and $K_3$ are pairwise disjoint. Let $W = \Sigma^n - \text{int}(N_1 \cup N_2)$, and choose embedded arcs $\alpha_1$ and $\alpha_2$ in $W$ from $q$ to points $q_1 \in \partial N_1$ and $q_2 \in \partial N_2$, respectively. Since $G_1$ is a normal subgroup of $\pi_1(\Sigma^n, q)$ (thus, invariant under conjugation), image($\pi_1(N_i \cup \alpha_i)$) = $G_i$ for $i = 1, 2$. Furthermore, since $K_i$ has codimension $\geq 3$, the inclusions $\Sigma^n - (K_1 \cup K_2) \subset \Sigma^n$ and $N_i - K_i \subset N_i$ ($i = 1, 2$) induce $\pi_1$-isomorphisms. Utilizing the collar structures on $N_i - K_i$, we may conclude that $W \subset \Sigma^n$ and $\partial N_i \subset N_i$ induce $\pi_1$-isomorphisms. By a slight abuse of notation, we write $\pi_1(W, q) = G_1 \times G_2$ with image($\pi_1(\partial N_i \cup \alpha_i, q)$) = $G_i$, $i = 1, 2$.

By (1) of §2, $\partial N_i$ and $\partial N_2$ are homology $(n-1)$-spheres; so, by [Ke, p. 71], there exist (combinatorial) compact contractible manifolds $C_1$ and $C_2$ with $\partial C_i \approx \partial N_i$ for each $i$. If $W \cup_\partial C_i$ denotes the space obtained by gluing $\partial C_i$ to $W$ along $\partial N_i$, VanKampen’s theorem gives an isomorphism $\pi_1(W \cup_\partial C_i, q) \to (G_1 \times G_2)/G_i$ for $i = 1, 2$. Furthermore, since the composition $G_i \to G_1 \times G_2 \to (G_1 \times G_2)/G_i$ is an isomorphism for $i = 1, 2$, we have inclusion induced isomorphisms, $\pi_1(K_3) \to \pi_1(W \cup_\partial C_i)$.

Reasoning as above, $\pi_1(W \cup_\partial (C_1 \cup C_2), q) \cong (G_1 \times G_2)/(G_1 \cup G_2) = \{1\}$.

Furthermore, by two applications of (3), $W \cup_\partial (C_1 \cup C_2)$ is a homology sphere. Hence, by the PL Generalized Poincaré Conjecture [Sm], $W \cup_\partial (C_1 \cup C_2) \approx S^n$.

Claim. $C_1$ and $C_2$ are fundamentally linked in $W \cup_\partial (C_1 \cup C_2) \approx S^n$.

Suppose there is an open cover $\{U_1, U_2\}$ of $W \cup_\partial (C_1 \cup C_2)$ by contractible sets with $C_i \subset U_i$ for $i = 1, 2$ and $C_i \cap U_j = \emptyset$ when $i \neq j$. Then $\{U_1 \cap K_3, U_2 \cap K_3\}$ is an open cover of $K_3$. By Lemma 3.3, we may assume without loss of generality that $U_1 \cap K_3$ contains a loop $\lambda$ which is nontrivial in $K_3$. Now, $U_1$ is contractible, so $\lambda$ contracts in $U_1 \subset W \cup_\partial C_1$. But, since $K_3 \subset W \cup_\partial C_1$ induces a $\pi_1$-isomorphism, this is impossible. □

Remark. In the above construction, the contractibility of $U_i$ was only used to assert that a loop $\lambda \subset U_i$ contracts in $U_i$. Hence, we have actually shown that $S^n$ cannot be covered by simply connected open sets $U_1$ and $U_2$ containing $C_1$ and $C_2$, respectively, and with $U_i \cap C_j = \emptyset$ for $i \neq j$.

Question. Does there exist a pair of fundamentally linked compact contractible polyhedra in $S^4$?

References


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