A NOTE ON THE DIFFERENTIABILITY OF CONVEX FUNCTIONS

WU CONGXIN AND CHENG LIXIN

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Abstract. Every real-valued convex and locally Lipschitzian function $f$ defined on a nonempty closed convex set $D$ of a Banach space $E$ is the local restriction of a convex Lipschitzian function defined on $E$. Moreover, if $E$ is separable and $\text{int} D \neq \emptyset$, then, for each Gateaux differentiability point $x \in \text{int} D$ of $f$, there is a closed convex set $C \subseteq \text{int} D$ with the nonsupport points set $N(C) \neq \emptyset$ and with $x \in N(C)$ such that $f_C$ (the restriction of $f$ on $C$) is Fréchet differentiable at $x$.


Theorem [Rainwater]. Suppose that $E$ is an Asplund space (a Banach space of class $(S)$). Then, for every closed convex subset $C$ of $E$ such that the nonsupport points set $N(C)$ of $C$ is nonempty and for every convex function $f$ on $C$ which is locally Lipschitzian on $N(C)$, the set of points $Q$, where $f$ is Fréchet (Gateaux) differentiable, is a dense $G_δ$ subset of $C$.

Notation. We denote by $C$, $C_x$, and $N(C)$ a closed convex set of a Banach space $E$, the cone generated by $C$ from $x$, and the nonsupport points set of $C$, respectively. The open and the closed balls centered at $x$ and with radius $r$ are denoted by $B(x, r)$ and $\overline{B}(x, r)$, respectively. All convex functions $f$, if nothing is added, are assumed real valued.

Definition. We say that a function denoted by $f_u$ is a local extension of $f$ at $u \in \text{dom}(f)$ and that $f$ is a local restriction of $f_u$ at $u$ provided there is a neighborhood $U$ of $u$ such that $f_u = f$ in $U \cap \text{dom}(f)$.

Definition. We say that $f$ is Gateaux (Fréchet) differentiable at $x \in N(C)$ provided there exists a unique $x^* \in E^*$ such that

$$
\langle x^*, y - x \rangle \leq f(y) - f(x) \quad \text{for all } y \in C
$$

(for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$
0 \leq f(y) - f(x) - \langle x^*, y - x \rangle \leq \varepsilon \|y - x\|
$$

whenever $y \in C \cap B(x, \delta))$. 

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Lemma 1. Suppose that \( f \) is convex and locally Lipschitzian on \( C \). Then
(a) for every \( u \in C \), there exist a neighborhood \( U \) of \( u \) and a convex Lipschitzian function \( f_u \) defined on \( E \) such that \( f = f_u \) in \( U \cap C \); 
(b) \( \text{dom}(\partial f) = \text{dom}(f) = C \); 
(c) there exists a selection \( \psi \) for \( \partial f \) on \( C \) such that \( \psi \) is locally bounded on \( C \); in particular \([1]\), if \( N(C) \neq \emptyset \), then \( \partial f \) is itself locally bounded on \( N(C) \).

Recall that the extended real-valued lower semicontinuous convex function \( f \) on \( E \) is said to be proper provided \( f(x) \geq -\infty \) for all \( x \in E \) and its (convex) essential domain
\[
\text{dom} f = \{ x : f(x) < +\infty \}
\]
is nonempty. For such \( f \), we define the inf-convolutions as
\[
f_n(x) = \inf \{ f(y) + n\| x - y \| : y \in E \}.
\]
Then we have (see, for instance, [3, 4, 6]).

Lemma 2. With \( f \) and \( \{ f_n \} \) as above, the sequence \( \{ f_n \} \) has the following properties:
(1) Each \( f_n \) is convex and Lipschitzian on \( E \) with Lipschitz constant \( n \); 
(2) \( f_n(x) \leq f_{n+1}(x) \leq f(x) \) for each \( x \in E \) and each \( n \geq 1 \); 
(3) \( f_n(x) = f(x) \) if and only if \( \partial f(x) \cap nB^* \) (where \( B^* \) denotes the unit ball of \( E^* \)) is nonempty or, equivalently, if and only if \( \partial f_n(x) = \partial f(x) \cap nB^* \).

Proof of Lemma 1. Since \( f \) is locally Lipschitzian on \( C \), there exists \( B(u, r) \) for some \( r > 0 \) and \( L > 0 \) such that \( |f(y) - f(x)| \leq L\| y - x \| \) whenever \( x, y \in B(u, r) \cap C = B_c(u, r) \). We define the extended real-valued convex function \( \hat{f} \) on \( E \) by \( \hat{f}(x) = f(x) \) if \( x \in B_c(u, r) \), and \( \hat{f}(x) = \infty \), otherwise. Hence \( f \) is proper lower semicontinuous convex and bounded below on \( E \) and Lipschitzian on \( B_c(u, r) \) with Lipschitz constant \( L \). Let \( \{ f_n \} \) be the sequence of the inf-convolutions by \( \hat{f} \). Now we will show (a). Suppose, to the contrary, that the neighborhood \( U \) of \( u \) does not exist. Let \( B_n = B_c(u, 1/n) \). Then, for each integer \( n \geq 1 \), there is \( x_n \in B_n \) such that \( f_n(x_n) < f(x_n) \), by the definition of \( f_n \). For each such \( x_n \), we can choose \( y_n \in E \) such that
\[
\hat{f}(x_n) - \hat{f}(y_n) > n\| y_n - x_n \|, \quad n = 1, 2, \ldots ,
\]
that is, \( \hat{f}(x_n) - \hat{f}(y_n) > n\| y_n - x_n \| \) for \( n = 1, 2, \ldots \). Clearly, we have \( y_n \in B_c(u, r) \). Note that \( x_n \in B_c(u, r) \) whenever \( 1/n < r \) and \( \hat{f} \) is Lipschitzian on \( B_c(u, r) \) with Lipschitz constant \( L \). We have \( |\hat{f}(x_n) - \hat{f}(y_n)| \leq L\| x_n - y_n \| \) whenever \( 1/n < r \); this is a contradiction which proved assertion (a).

From the proof of assertion (a), it is easy to see that \( f_n = \hat{f} \) in \( B_c(u, r) \) by taking \( n = [L] + 1 \) (where \([L]\) denotes the maximal integer \( m \) satisfying \( m \leq L \)); that is, by Lemma 2, \( \partial \hat{f}(x) \cap nB^* \) is nonempty for each \( x \in B_c(u, r) \). Hence \( \partial f(x) \cap nB^* \) is nonempty for each \( x \in B(u, r) \). Now we showed that there is a selection for \( \partial f \) on \( B(u, r) \) which is bounded by \( n \). The arbitrariness of \( u \) says that there is a selection \( \psi \) for \( \partial f \) on \( C \) which is locally bounded; hence we proved assertions (b) and (c).

With \( f \) and \( f_u \) as in Lemma 1, by Lemma 2, we have \( \partial f_u(x) \subset \partial f(x) \) for each \( x \in U \cap C \). If \( C \) is closed and \( N(C) \neq \emptyset \), note that \( x \in N(C) \) if
and only if $C_x$ is dense in $E$ [1]. By a simply convexity and differentiability argument, we see $\partial f = \partial f_u$ in $U \cap N(C)$. Hence we have

**Proposition 3.** Suppose that $C$ is closed with $N(C) \neq \emptyset$ and $f$ is locally Lipschitzian on $N(C)$. Then the following versions are equivalent:

(a) $f$ is Gateaux differentiable at $u \in N(C)$.
(b) Every $f_u$ is Gateaux differentiable at $u \in N(C)$.
(c) Every selection for $\partial f$ on $N(C)$ is norm-to-weak* continuous at $u$.
(d) There is a selection for $\partial f$ on $N(C)$ which is norm-to-weak* continuous at $u$.

**Corollary 4.** Suppose that $C$ is a closed convex set of the weak Asplund space $E$, and suppose that the convex function $f$ is locally Lipschitzian on $N(C)$. Then the set of points $G$, where $f$ is Gateaux differentiable, is a $G_\delta$ set of $C$.

It is easy to see that if $u \in N(C)$ is the Fréchet differentiability point of $f_u$, then so is the one of $f$. The following example shows that if the interior of $C$ is empty, then the Fréchet differentiability of $f_u$ is really stronger than the one of $f$.

**Example 5.** The function $f$ defined by $f(x) = \|x\|_1$ on $C = \{x \in l^1, x_n \geq 0$ for $n = 1, 2, \ldots\}$ shows that $f$ is Fréchet differentiable at each point of $N(C) = \{x \in l^1, x_n > 0$ for $n = 1, 2, \ldots\}$, but the extension $f_u$ of $f$ (where $f_u = \|\cdot\|_1$ on $l^1$) is nowhere Fréchet differentiable.

More generally, we have

**Theorem 6.** Suppose that $E$ is separable, $D$ is a nonempty open convex set of $E$, and $f$ is a continuous convex function on $D$. Then, for each Gateaux differentiability point $x \in D$ of $f$, there exists a closed convex subset $C$ of $D$ with $N(C) \neq \emptyset$ and $x \in N(C)$ such that $f_C$ (the restriction of $f$ to $C$) is Fréchet differentiable at $x$.

**Proof.** Let $0 < \varepsilon < \frac{1}{2}$. By [8] there are two sequences $\{x_j\}$ and $\{x_j^*\}$ in $E$ and in $E^*$, respectively, satisfying

1. $\|x_j\| = 1$ for $j = 1, 2, \ldots$;
2. $\|x_j^*\| < 1 + \varepsilon$ for $j = 1, 2, \ldots$;
3. $x_j^*(x_j) = \delta_{ij}$ ($= 1$ if $i = j$, $= 0$ otherwise);
4. $\text{cl}(\text{span}(x_j \text{ for } j = 1, 2, \ldots)) = E$.

Suppose that $x \in D$ is a Gateaux differentiability point of $f$. Then, for each $n \geq 1$, there exists $1 \geq \delta_n > 0$ ($\delta_n \to 0$) such that

$$f(x \pm 2\delta_n x_n) - f(x) - \langle x^*, \pm 2\delta_n x_n \rangle < 2^{-n} \quad (x^* = \partial f(x)).$$

Let $E_n = \text{span}(x_j \text{ for } j = 1, 2, \ldots, n)$, $F = \overline{\text{co}}(\pm y_n \text{ for } n = 1, 2, \ldots)$, and $C = x + F$, where $y_n = \delta_n x_n$ for $n = 1, 2, \ldots$. Clearly, $C (\subset D)$ is closed and convex and $C_x$ is dense in $E$, and hence $x \in N(C)$. Now we prove that point $x$ is a Fréchet differentiability point of $f_c$. Suppose, to the contrary, that $f_c$ is not Fréchet differentiable at $x$. Then there is $\varepsilon_0 > 0$ such that for each
there is \( z_n \in C \) with \( 0 < \| z_n - x \| < \delta_n \) such that

\[
\frac{f(c(z_n)) - f(c(x)) - (x^*, z_n - x)}{\| z_n - x \|} \geq \varepsilon_0 \quad (n = 1, 2, \ldots),
\]

where \( x^* = \partial f(x) \). By the density of \( co(\pm y_n) \) for \( n = 1, 2, \ldots \) in \( F \), we can assume that \( \{z_n - x\} \subset co(\pm y_n) \) for \( n = 1, 2, \ldots \), that is, for each \( n \geq 1 \), there is a sequence \( \{\lambda_i^{(n)}\} \) with \( \lambda_i^{(n)} \geq 0 \) for \( i = 1, 2, \ldots \) and \( \sum_i \lambda_i^{(n)} = 1 \) such that \( z_n - x = \sum_i \lambda_i^{(n)}(\pm y_i) \). Therefore, we have, by the properties of \( \{x_n\} \) and \( \{x_n^*\} \),

\[
\| z_n - x \| > \frac{1}{1 + \varepsilon} \max_j |x_j^*(z_n - x)| = \frac{1}{1 + \varepsilon} \max_j \lambda_j^{(n)} \| y_j \| = \frac{1}{1 + \varepsilon} \max_j \lambda_j^{(n)} \delta_j.
\]

Fix \( n_0 \) such that \( (1 + \varepsilon) \sum_{j=n_0+1}^{\infty} 2^{-j} = (1 + \varepsilon)2^{-n_0} < \varepsilon_0/2 \). Let

\[
u_n = x + 2 \sum_{i=1}^{n_0} \lambda_i^{(n)}(\pm y_i) \quad \text{and} \quad \nu_n = x + 2 \sum_{i=n_0+1}^{\infty} \lambda_i^{(n)}(\pm y_i).
\]

Since \( z_n \to x \), we can claim \( u_n, v_n \in D \) for \( n = 1, 2, \ldots \). Now we have \( z_n = (u_n + v_n)/2 \), therefore,

\[
f_c(z_n) = f(z_n) = f((u_n + v_n)/2) \leq \frac{1}{2}[f(u_n) + f(v_n)].
\]

Let \( w_n = (u_n - x)/\theta_n \) (where \( \theta_n = \| z_n - x \| ) \). Then

\[
\| w_n \| = \| u_n - x \| / \theta_n \leq \sum_{i=1}^{n_0} 2\lambda_i^{(n)} \| y_i \| / \theta_n = 2 \sum_{i=1}^{n_0} \lambda_i^{(n)} \delta_i / \theta_n
\]

\[
\leq (1 + \varepsilon) \cdot 2 \sum_{i=1}^{n_0} \lambda_i^{(n)} \delta_i / \max_j \lambda_j^{(n)} \delta_j \leq 2(1 + \varepsilon)n_0;
\]

that is, \( \{w_n\} \) is contained by \( 2(1 + \varepsilon)n_0B_{n_0} \), where \( B_{n_0} \) is the unit ball of the \( n_0 \)-dimensional subspace \( E_{n_0} \) of \( E \).

\[
f(v_n) = f \left( x + 2 \sum_{i=n_0+1}^{\infty} \lambda_i^{(n)}(\pm y_i) \right) = f \left( \sum_{i=n_0+1}^{\infty} \lambda_i^{(n)}(x \pm 2y_i) + \sum_{i=1}^{n_0} \lambda_i^{(n)}x \right)
\]

\[
\leq \sum_{i=n_0+1}^{\infty} \lambda_i^{(n)} f(x \pm 2y_i) + \sum_{i=1}^{n_0} \lambda_i^{(n)} \cdot f(x).
\]
Hence

\[ \varepsilon_0 \leq \frac{f(z_n) - f(x) - \langle x^*, z_n - x \rangle}{\theta_n} \]

\[ \leq \frac{\{[f(u_n) + f(v_n)]/2 - f(x) - \langle x^*, z_n - x \rangle\}}{\theta_n} \]

\[ = \frac{f(u_n) - f(x) - \langle x^*, u_n - x \rangle + f(v_n) - f(x) - \langle x^*, v_n - x \rangle}{2\theta_n} \]

\[ \leq \frac{f(u_n) - f(x) - \langle x^*, u_n - x \rangle}{2\theta_n} \]

\[ + \sum_{i=n_0+1}^{\infty} \lambda_i^{(n)}(f(x \pm 2y_i) - f(x) - \langle x^*, \pm 2y_i \rangle) \]

\[ \leq \frac{f(u_n) - f(x) - \langle x^*, u_n - x \rangle}{2\theta_n} \]

\[ + \frac{(1 + \varepsilon) \sum_{i=n_0+1}^{\infty} \lambda_i^{(n)}(f(x \pm 2y_i) - f(x) - \langle x^*, \pm 2y_i \rangle)}{2 \max_j \lambda_j^{(n)} \delta_j} \]

\[ \leq \frac{f(u_n) - f(x) - \langle x^*, u_n - x \rangle}{2\theta_n} \]

\[ + \frac{(1 + \varepsilon) \sum_{j=n_0+1}^{\infty} 2^{-j}}{2 \max_j \lambda_j^{(n)} \delta_j} \]

\[ < \frac{f(u_n) - f(x) - \langle x^*, u_n - x \rangle}{2\theta_n} \]

\[ + \frac{\varepsilon_0}{2} \]

\[ = \frac{f(x + \theta_n w_n) - f(x) - \langle x^*, \theta_n w_n \rangle}{\theta_n} + \frac{\varepsilon_0}{2} \]

Since \(2(1 + \varepsilon)n_0B_{n_0}\) is a bounded set of the \(n_0\)-dimensional subspace \(E_{n_0}\) of \(E\), \(w_n \in 2(1 + \varepsilon)n_0B_{n_0}\), and \(\theta_n \to 0\), and since \(f\) is Gateaux differentiable at \(x\), we must have

\[ \frac{f(x + \theta_n w_n) - f(x) - \langle x^*, \theta_n w_n \rangle}{\theta_n} < \varepsilon_0 \]

for sufficiently large \(n\). Hence

\[ \varepsilon_0 \leq \frac{f(x + \theta_n w_n) - f(x) - \langle x^*, \theta_n w_n \rangle}{2\theta_n} + \frac{\varepsilon_0}{2} < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0 \]

for sufficiently large \(n\), and this is a contradiction which completes our proof.

Remark. The recent work of Preiss, Phelps, and Namioka [7] showed that if \(E\) admits an equivalent smooth norm, then it is of class \((S)\). Hence, for the Gateaux differentiability, the Rainwater theorem still holds for smoothable Banach spaces.

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Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, People’s Republic of China