SIEGEL'S THEOREM FOR COMPLEX FUNCTION FIELDS

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Abstract. We give a short proof of the finiteness of the set of integral points on an affine algebraic curve of genus at least one, defined over a function field of characteristic zero.

Siegel [Si] has shown that an affine algebraic curve of genus at least one defined over a number field has only finitely many integral points. Lang [L] has proven an analogous result for curves defined over a function field of characteristic zero not defined over the constant field. For curves of genus at least two, one even has the Mordell conjecture (proved by Faltings [F] in the number field case and by Manin [M] in the function field case) that there are only finitely many rational points.

For genus one, Manin [M] gave a proof of a strengthening of Lang's result as a by-product of his work on the Mordell conjecture. Mason [Ms] then gave an effective proof by more elementary considerations. In this note, we give a short proof of Manin's (and hence Lang's) result for genus one. The proof can be adapted to higher genus as well (see the remark below).

Let $K$ be a function field with constant field of characteristic zero and $E/K$ an elliptic curve with nonconstant $j$-invariant. The reader may consult [S] for definitions and results about elliptic curves. In particular, we shall use the following results. The group $E(K)$ is a finitely generated abelian group by the Mordell-Weil theorem, and there is a height function $h : E(K) \to \mathbb{R}$ with the property that there are only finitely many points of bounded height ([IL], Proposition 2). The height can be written as a sum of local heights $\sum \lambda_v(P)$, where $v$ ranges through the places of $K$. The local heights satisfy $\lambda_v(P) = \max\{0, v(t(P))\} + \beta_v(P)$, where $\beta_v$ is bounded for all $v$ and is identically zero for all but finitely many $v$, and $t$ is a uniformizer at $0 \in E$. For example, $t = x/y$, where $x, y$ are coordinates of a Weierstrass equation for $E$.

Now, Lang's result for genus one can be reduced to the case of a Weierstrass equation ([S], Corollary IX.3.2.2) and in this case we argue as follows. Let $S$ be a finite set of places of $K$. Then $\sum_{\notin S} \lambda_v(P)$ is bounded independently of $P$, if $P$ is $S$-integral and, since there are only finitely many points of bounded
height, if there are infinitely many $S$-integral points, then $\lambda_v$ is unbounded for some $v \in S$. It suffices thus to prove the following result:

**Theorem (Manin).** Let $K$ be a function field with constant field of characteristic zero and $E/K$ an elliptic curve with nonconstant $j$-invariant and $v$ a place of $K$. Then the local height function $\lambda_v$ is bounded on $E(K)$.

**Proof.** The points on $E(K_v)$ that reduce to $0 \bmod v$ form a subgroup $E_1(K_v)$ isomorphic to the group of points of a formal group. Choosing an uniformizer $t$, as above, on $E$ at $0$, then $E_1(K_v) = \{ P \in E(K_v) \mid t(P) \in \mathcal{M}_v \}$, where $\mathcal{M}_v$ is the maximal ideal of the local ring at $v$. Moreover, $\lambda_v(P)$ differs from $v(t(P))$ by a bounded amount. Hence, it suffices to show that $v(t(P))$ is bounded above on $E(K)$. Suppose not and choose $P_n, n = 1, 2, \ldots, \in E(K)$ such that $v(t(P_{n+1})) > v(t(P_n)) > 0$. We claim that $P_1, P_2, \ldots$ are linearly independent over $Z$. Recall that $t$ induces a group isomorphism between $E_r/E_{r+1}$, where $E_r = E_r(K_v) = \{ P \in E(K_v) \mid t(P) \in \mathcal{M}_v \}$, and $\mathcal{M}_v / \mathcal{M}_v^{r+1}$. If $n_1 P_1 = \sum_{j \geq 1} n_j P_j$, $n_i \neq 0$, and $r = v(t(P_i))$ then $n_i P_i$ is $0$ in $E_r/E_{r+1}$, but $t(n_1 P_1) \equiv n_1 t(P_i) \not\equiv 0 \pmod{\mathcal{M}_v^{r+1}}$, which proves the claim. On the other hand, the claim contradicts the Mordell-Weil theorem and this completes the proof.

**Remark.** On a curve of genus greater than one, if a sequence of points $P_1, P_2, \ldots$ approaches rapidly a point $P_\infty$, then a similar argument shows that the $P_i - P_\infty$ are linearly independent over $Z$ in the Jacobian of the curve, and Lang's result follows from this. The author and A. Buium [BV] have recently proved a conjecture of Lang to the effect that an affine open subset of an abelian variety of any dimension over a function field of characteristic zero has finitely many integral points.

**References**


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