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SCHARLEMANN'S 4-MANIFOLDS AND SMOOTH 2-KNOTS IN $S^2 \times S^2$

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ABSTRACT. Scharlemann gave an example of a 4-manifold admitting a fake homotopy structure on $S^3 \times S^1 \sharp S^2 \times S^2$, which is homeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2$ by a theorem of Freedman. We address the problem whether a Scharlemann's manifold is diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2$ in terms of 2-knots in $S^2 \times S^2$.

1. INTRODUCTION

Since Milnor's discovery [18] of exotic 7-spheres which are homeomorphic to the standard 7-sphere but not diffeomorphic, the existence of exotic smooth structures in dimension 5 and higher has been well known. Meanwhile, in dimension 3 and less, there are no exotic smooth structures; that is, homeomorphic smooth manifolds are diffeomorphic.

The first example of an exotic smooth structure in dimension 4 is the fake $\mathbb{R}P^4$ of Cappell and Shaneson [2], which is homeomorphic to $\mathbb{R}P^4$ by a theorem of Freedman [8] but not diffeomorphic. This manifold, however, is nonorientable.

As for orientable 4-manifolds, in conjunction with Freedman's work [7], Donaldson's theorem [3] on the nonrepresentation of any nontrivial definite form by a smooth 4-manifold gives the remarkable result that $\mathbb{R}^4$ admits an exotic smooth structure [12]. Furthermore, Donaldson gave, in [4], the first example of exotic orientable closed 4-manifolds by using the Γ-invariant introduced by him: a certain Dolgachev surface is an exotic $\mathbb{C}P^2 \sharp 9\,\overline{\mathbb{C}P^2}$. The Dolgachev surface and $\mathbb{C}P^2 \sharp 9 \overline{\mathbb{C}P^2}$ are h-cobordant nondiffeomorphic manifolds, so they give a counterexample to the h-cobordism conjecture for simply connected smooth 4-manifolds. Other related exotic manifolds have been found by using the Γ-invariant or another guage theoretic invariant [9, 10].

Before these discoveries in the 1980s of exotic orientable closed 4-manifolds, Scharlemann found a family of strange orientable closed 4-manifolds in 1976 [25]. Every Scharlemann's manifold gives a fake homotopy structure for $S^3 \times S^1 \sharp S^2 \times S^2$. Meanwhile, since a Scharlemann's manifold and $S^3 \times S^1 \sharp S^2 \times S^2$ are

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s-cobordant, the 5-dimensional topological s-cobordism theorem for $\pi_1 \cong \mathbb{Z}$ proved by Freedman [8] states that they are homeomorphic. From a Scharlemann's manifold and $S^3 \times S^1 \sharp S^2 \times S^2$, Gompf constructed two compact orientable 4-manifolds with boundary which are homeomorphic but not diffeomorphic [14]. Then a question comes to mind.

Problem (Matsumoto [15, Problem 4.15]). Is a Scharlemann's manifold diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2$?

This problem asks whether a Scharlemann's manifold gives an exotic structure on $S^3 \times S^1 \sharp S^2 \times S^2$ and whether they give a counterexample to the s-cobordism conjecture for smooth 4-manifolds with $\pi_1 \cong \mathbb{Z}$. In this note, we address this problem. We note that after connected sum with $S^2 \times S^2$ a Scharlemann's manifold becomes diffeomorphic to $S^3 \times S^1 \sharp 2(S^2 \times S^2)$, which Fintushel and Pao proved [6]. (In fact, Gompf proved that compact orientable homeomorphic 4-manifolds must become diffeomorphic after connected sum with an unspecified number of copies of $S^2 \times S^2$ [13].)

We show that for each Scharlemann's manifold X there exists a null-homologous smooth 2-knot in $S^2 \times S^2$ which is topologically trivial and has the following property: This 2-knot is smoothly trivial if and only if X is diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2$.

Also, by using our 2-knots one may show that after connected sum of the twisted S^2-bundle $S^2 \tilde{\times} S^2$ over S^2 a Scharlemann's manifold becomes diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2 \sharp S^2 \tilde{\times} S^2$.

2. Scharlemann's manifolds and 2-knots in $S^2 \times S^2$

For a smooth manifold M and a submanifold A of M, we denote a tubular neighborhood of A in M by $N(A)$. In this note, we call a smoothly embedded 2-sphere in $S^2 \times S^2$ a 2-knot in $S^2 \times S^2$.

Let K be a 2-knot in S^4 with exterior $E(K)$ and C a smooth circle in S^4 disjoint from K. Since we may assume that C is standardly embedded in S^4 up to ambient isotopy, K is contained in $S^4 - \operatorname{int} N(C) = S^2 \times D^2_+$. This gives a 2-knot in $S^2 \times S^2 = S^2 \times D^2_+ \cup S^2 \times D^2_-$ and is denoted by $S(K, C)$. It follows from van Kampen's theorem that the knot group of $S(K, C)$, $\pi_1(S^2 \times S^2 - S(K, C))$, is isomorphic to $\pi_1(S^4 - K)/H$, where H is the normal closure generated by the element represented by C in $\pi_1(S^4 - K)$ [16, 23, 24].

Definition. Let S be a 2-knot in $S^2 \times S^2$. Then S is said to be smoothly (topologically, resp.) trivial if S bounds a smooth (topological, resp.) 3-ball in $S^2 \times S^2$.

We consider a nontrivial fibered 2-knot K in S^4 with closed fiber M. Suppose that M is a homology 3-sphere such that $\pi_1(M)$ has weight one. Then the exterior $E(K)$ of K is a fiber bundle over S^1 with fiber M° and monodromy map $\sigma : M^\circ \to M^\circ$; i.e.,

$$E(K) = M^\circ \times_\sigma S^1 = M^\circ \times I/(x, 0) \sim (\sigma(x), 1),$$

where M° is a punctured copy of M. Let α be a weight element for $\pi_1(M^\circ) = \pi_1(M)$. We take a smooth circle in M° representing α and denote the circle by the same symbol α. (For a manifold X, we shall sometimes not distinguish notationally between an element of $\pi_1(X)$ and a smooth circle in X representing it.) Let α_* denote a smooth circle $\alpha \times \{*\}$ on a fiber $M^\circ \times \{*\}$ in $E(K) = M^\circ \times_\sigma S^1$. Then $S(K, \alpha_*)$ is a null-homologous 2-knot in $S^2 \times S^2$ satisfying

$$\pi_1(S^2 \times S^2 - S(K, \alpha_*)) \cong \mathbb{Z} \cong \pi_1(S^2 \times S^2 - \text{trivial 2-knot}).$$

In fact, the knot group $\pi_1(S^4 - K)$ has the presentation

$$\langle\ \pi_1(M^\circ),\ \mu \mid \mu x \mu^{-1} = \sigma_\sharp(x) \text{ for any } x \in \pi_1(M^\circ)\ \rangle.$$

Since $\alpha \in \pi_1(M^\circ)$ is a weight element, $\pi_1(S^2 \times S^2 - S(K, \alpha_*)) \cong \langle \mu | \ \rangle \cong \mathbb{Z}$. The 2-knot $S(K, \alpha_*)$ has a tubular neighborhood diffeomorphic to $S^2 \times D^2$, and so the self-intersection number of $S(K, \alpha_*)$ in $S^2 \times S^2$ is zero. Hence, $S(K, \alpha_*)$ represents $p\zeta$ or $p\eta$ for some integer p, where ζ and η are natural generators of $\mathrm{H}_2(S^2 \times S^2; \mathbb{Z})$ with $\zeta \cdot \zeta = \eta \cdot \eta = 0$ and $\zeta \cdot \eta = \eta \cdot \zeta = 1$. Meanwhile, since $\mathrm{H}_1(S^2 \times S^2 - S(K, \alpha_*); \mathbb{Z}) \cong \pi_1(S^2 \times S^2 - S(K, \alpha_*)) \cong \mathbb{Z}$, it follows that $p = 0$; namely, $S(K, \alpha_*)$ is null-homologous.

Next we consider Scharlemann's manifolds. These manifolds are constructed as follows: Let Σ be the Poincaré homology 3-sphere, which is the intersection of the unit 5-sphere in $\mathbb{C}^3$ with the complex variety $z_0^2 + z_1^3 + z_2^5 = 0$. The fundamental group $\pi_1(\Sigma)$ is the binary dodecahedral group and has weight one. Let α be a weight element. Let α_* denote a smooth circle on a fiber $\Sigma \times \{*\}$ in $\Sigma \times S^1$ representing $\alpha \in \pi_1(\Sigma) \triangleleft \pi_1(\Sigma \times S^1)$. Remove a tubular neighborhood of α_* in $\Sigma \times S^1$, and attach $S^2 \times D^2$ to its boundary with the trivial framing. The resulting manifold X_α obtained from $\Sigma \times S^1$ by this surgery is a *Scharlemann's manifold*, which is a closed orientable smooth 4-manifold with $\pi_1(X_\alpha) \cong \mathbb{Z}$. The Poincaré homology 3-sphere Σ is smoothly embedded in S^5 as the link of the algebraic variety defined above. Then $\pi_1(S^5 - \Sigma) \cong \mathbb{Z}$, and by Alexander duality $S^5 - \Sigma$ is a homology circle. Thus, by doing surgery of $S^5 - \operatorname{int} N(\Sigma)$, we can obtain an h-cobordism W between X_α and $S^3 \times S^1 \sharp S^2 \times S^2$. Furthermore, since the Whitehead group $Wh(\mathbb{Z}) = 0$, W is a s-cobordism between them. Hence, we have

Lemma 1 [25]. *Two manifolds X_α and $S^3 \times S^1 \sharp S^2 \times S^2$ are s-cobordant.*

We relate Scharlemann's manifolds to certain null-homologous 2-knots in $S^2 \times S^2$:

Proposition 2. *For any weight element $\alpha \in \pi_1(\Sigma)$, there exists a null-homologous 2-knot S_α in $S^2 \times S^2$ satisfying:*

(1) *S_α is topologically trivial in $S^2 \times S^2$ and*
(2) *S_α is smoothly trivial in $S^2 \times S^2$ if and only if the Scharlemann's manifold X_α is diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2$.*

Such 2-knots are constructed as follows: Let β be a weight element for $\pi_1(\Sigma)$. In $\Sigma \times S^1$ let βt be a simple loop representing β times t in $\pi_1(\Sigma \times S^1)$ such that βt meets each $\Sigma \times \{*\}$ transversely in a single point, where t is an element represented by $\{*\} \times S^1 \subset \Sigma \times S^1$. Then $E = \Sigma \times S^1 - \operatorname{int} N(\beta t)$ is a fiber bundle over S^1 with fiber Σ°. We consider a pair of spaces

$$(Y, K_\beta) = (E \cup_{\partial E} S^2 \times D^2, S^2 \times \{0\}).$$

Since βt is a weight element for $\pi_1(\Sigma \times S^1)$, Y is a homotopy 4-sphere. In particular, if some power of β lies in the center $Z(\pi_1(\Sigma))$ of $\pi_1(\Sigma)$, then the monodromy map σ has finite order, so Y is a diffeomorphic to S^4 [19, 21, 22]. Now we take such an element β. Thus K_β is a fibered 2-knot in S^4 with exterior $E(K_\beta) = \Sigma \times S^1 - \operatorname{int} N(\beta t) = \Sigma^\circ \times_\sigma S^1$. It follows that, for every weight element $\alpha \in \pi_1(\Sigma)$, $S(K_\beta, \alpha_*)$ is a null-homologous 2-knot in $S^2 \times S^2$ with $\pi_1(S^2 \times S^2 - S(K_\beta, \alpha_*)) \cong \mathbb{Z}$. Let $E(S(K_\beta, \alpha_*))$ be the exterior of $S(K_\beta, \alpha_*)$ in $S^2 \times S^2$. We now consider the closed orientable smooth 4-manifold $M(S(K_\beta, \alpha_*))$ obtained from $S^2 \times S^2$ by surgery along $S(K_\beta, \alpha_*)$, i.e., $M(S(K_\beta, \alpha_*)) = D^3 \times S^1 \cup_{\partial E(S(K_\beta, \alpha_*))} E(S(K_\beta, \alpha_*))$. Noting that the surgery on βt in $\Sigma \times S^1$ and the surgery on K_β in S^4 simply undo each other, it follows that $M(S(K_\beta, \alpha_*))$ is diffeomorphic to X_α. Our construction may be described by:

$$
\begin{tikzcd}
\Sigma \times S^1 \arrow[rr, "\text{surger } \beta t"] \arrow[dd, "\text{surger } \alpha_*"'] & & S^4 \supset K_\beta \arrow[dd, "\text{surger } \alpha_*"] \arrow[ll, shift left=8, "\text{surger } K_\beta"'] \\
& & \\
X_\alpha & & S^2 \times S^2 \supset S(K_\beta, \alpha_*) \arrow[ll, "\text{surger } S(K_\beta, \alpha_*)"]
\end{tikzcd}
$$

Proposition 2 then follows from the following lemma.

Lemma 3. *Let S be a null-homologous 2-knot in $S^2 \times S^2$ such that $\pi_1(S^2 \times S^2 - S) \cong \mathbb{Z}$. Let X be the resultant 4-manifold of surgery along S. Then X is diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2$ if and only if S bounds a smooth 3-ball in $S^2 \times S^2$.*

Proof. It is clear that if S bounds a smooth 3-ball in $S^2 \times S^2$, then X is diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2$.

Suppose that X is diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2$. Then there exists a diffeomorphism $f : X \to S^3 \times S^1 \sharp S^2 \times S^2$. Let D^3_+ and D^3_- be 3-balls such that $S^3 = D^3_+ \cup D^3_-$ and $D^3_+ \cap D^3_- = \partial D^3_+ = \partial D^3_-$. We may assume that the 4-ball in $S^3 \times S^1$ of the connected sum $S^3 \times S^1 \sharp S^2 \times S^2$ lies in the interior of $D^3_- \times S^1 \subset S^3 \times S^1$. Let $E(S)$ be the exterior of S. Then we have two smooth circles $C_0 = \{0\} \times S^1 \subset D^3 \times S^1 \subset D^3 \times S^1 \cup_{\partial E(S)} E(S) = X$ and $C_1 = \{0_+\} \times S^1 \subset D^3_+ \times S^1 \subset (D^3_+ \cup D^3_-) \times S^1 \sharp S^2 \times S^2 = S^3 \times S^1 \sharp S^2 \times S^2$ such that $N(C_0) = D^3 \times S^1 \subset D^3 \times S^1 \cup_{\partial E(S)} E(S) = X$ and $N(C_1) = D^3_+ \times S^1 \subset S^3 \times S^1 \sharp S^2 \times S^2$. The homotopy classes of C_0 and of C_1 generate $\pi_1(X) \cong \mathbb{Z}$ and $\pi_1(S^3 \times S^1 \sharp S^2 \times S^2) \cong \mathbb{Z}$ respectively. Since the diffeomorphism $f : X \to S^3 \times S^1 \sharp S^2 \times S^2$ induces an isomorphism $f_\sharp : \mathbb{Z} \cong \langle [C_0] \rangle \to \langle [C_1] \rangle \cong \mathbb{Z}$, $f_\sharp([C_0]) = [C_1]^{\pm 1}$. Hence, two smooth circles $f(C_0)$ and C_1 are free homotopic in $S^3 \times S^1 \sharp S^2 \times S^2$. Since $\dim f(C_0) = \dim C_1 = 1$ and $\dim(S^3 \times S^1 \sharp S^2 \times S^2) = 4$, there exists an ambient isotopy $\{h_t : S^3 \times S^1 \sharp S^2 \times S^2 \to S^3 \times S^1 \sharp S^2 \times S^2\}_{t \in I}$ such that $h_1(f(C_0)) = C_1$. Thus the diffeomorphism $f' = h_1 \circ f : X \to S^3 \times S^1 \sharp S^2 \times S^2$ takes C_0 to C_1. Moreover, deforming f' via an ambient isotopy of $S^3 \times S^1 \sharp S^2 \times S^2$ if necessary, we obtain a diffeomorphism $\widetilde{f} : X \to S^3 \times S^1 \sharp S^2 \times S^2$ such that $\widetilde{f}(N(C_0)) = N(C_1)$ by the uniqueness of tubular neighborhoods. Therefore, the restriction of $\widetilde{f}$ to $E(S)$ gives a diffeomorphism

$$g : E(S) \to D^3_- \times S^1 \sharp S^2 \times S^2.$$

Identifying the boundaries of $E(S)$ and $D^3_- \times S^1 \sharp S^2 \times S^2$ with $S^2 \times S^1$, we may assume, by [11], that, for some point $* \in S^1$, $g(S^2 \times \{*\}) = S^2 \times \{*\}$. Hence, the 3-ball $g^{-1}(D^3_- \times \{*\})$ in $E(S)$ provides a 3-ball in $S^2 \times S^2$ bounded by S. This implies that S is smoothly trivial in $S^2 \times S^2$. □

Thus $S(K_\beta, \alpha_*)$ satisfies property (2).

By Lemma 1, two manifolds X_α and $S^3 \times S^1 \sharp S^2 \times S^2$ are s-cobordant, so it follows from the 5-dimensional topological s-cobordism theorem for $\pi_1 \cong \mathbb{Z}$ [8] that they are homeomorphic, that is, there is a homeomorphism $g : X_\alpha \to S^3 \times S^1 \sharp S^2 \times S^2$. Hence $S(K_\beta, \alpha_*)$ is topologically trivial in $S^2 \times S^2$ for a similar reason as above. This completes the proof of Proposition 2. □

Remark. Proposition 2 implies that if a Scharlemann's manifold X_α is not diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2$, then the 2-knot S in Proposition 2 will give an exotic knotting of the trivial 2-knot in $S^2 \times S^2$. Any example of exotic knottings of S^2 into a 4-manifold has not been known yet, but some examples of exotic knottings of 2-disc or nonorientable surfaces into a 4-manifold are known [5, 26].

Matumoto proved in [17] that for a smooth simply connected 4-manifold Y if S is a null-homologous 2-knot in Y with $\pi_1(Y - S) \cong \mathbb{Z}$, then S is smoothly trivial in $Y \sharp n(S^2 \times S^2)$ for some nonnegative integer n; namely, S bounds a smooth 3-ball in it. Hence, it follows from Proposition 2 that the 2-knot S_α is smoothly trivial in $S^2 \times S^2 \sharp n(S^2 \times S^2)$ for some nonnegative integer n and $X_\alpha \sharp n(S^2 \times S^2)$ is diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2 \sharp n(S^2 \times S^2)$.

Fintushel and Pao proved that $X_\alpha \sharp S^2 \times S^2$ is diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2 \sharp S^2 \times S^2$ [6], so the 2-knot S_α is smoothly trivial in $S^2 \times S^2 \sharp S^2 \times S^2$. In a use of a 2-knot S_α in $S^2 \times S^2$ one may show that, for the twisted S^2-bundle $S^2 \widetilde{\times} S^2$ over S^2, $X_\alpha \sharp S^2 \widetilde{\times} S^2$ is diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2 \sharp S^2 \widetilde{\times} S^2$. But this follows from a better result proved by Akbulut [1]: $X_\alpha \sharp \mathbb{C}P^2$ is diffeomorphic to $S^3 \times S^1 \sharp S^2 \times S^2 \sharp \mathbb{C}P^2$.

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