CORRIGENDUM TO
"ON HAUSDORFF DIMENSION OF RECURRENT NET FRACTALS"

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(Communicated by Kenneth Meyer)

In the proof of Theorem 3.1 in [6] a characterization of Larma's finite-dimen-
sional metric spaces due to Rogers [5] was taken for granted and used. Since this characterization is not true in general, in this note we add a further hypothesis on the complete metric space treated in [6] which guarantees the validity of Theorem 3.1. Moreover, some natural conditions tacitly assumed in [6] are made explicit, thus extending the geometric analysis. Proposition 1.1 is correctly stated and improved.

By defining in [6] net fractals in a complete metric space, we intended to provide a procedure for generating sets which look like fractals. They are expected to be, topologically, at least perfect subsets of the given metric space and in particular uncountable sets. In [6] we, tacitly, assumed (without explicitly stating) that

\[ \mathcal{N} \cap \text{int}(A_{\lfloor n \rfloor}) \neq \emptyset \quad \forall \lfloor n \rfloor, \]

which guarantees that \( \mathcal{N} \) has the above topological properties. Further (A) is a natural condition and necessary to avoid that the geometric procedure described in [6] collapses. In fact, if (A) is not assumed the set \( \mathcal{N} \) might be contained in the boundary set \( B := \bigcup_{\lfloor n \rfloor} A_{\lfloor n \rfloor} \setminus \text{int}(A_{\lfloor n \rfloor}) \); thus, by standard results (see [3]), in that case we will have \( \mathcal{H}^s_d(\mathcal{N}) = 0 \), for any dimension \( s \) for which \( \mathcal{H}^s_d(\mathcal{N}) < \infty \) (here \( \mathcal{H}^s_d \) and \( \mathcal{H}^s_D \) represent the Hausdorff measures with respect to the metrics \( d \) and \( D \) used in [6]). Therefore, conditions (2) and (3) in [6] would become insignificant within the scope of the dimension estimate of \( \mathcal{N} \).

See also the discussion about net fractals generated by 'proper constructions' in [1], which is the reference given in [6] for the original definition.

An analogous assumption was made in [6, §4] for the self-similar set \( K \). It is natural to assume, as it happens in most concrete examples, that \( K \) is not completely contained in the boundary of the open set \( O \), for otherwise, \( K \) would be the kernel of a geometric scheme does not satisfy condition (A); consequently, by the above remarks, it would be \( \mathcal{H}^s_d(K) = 0 \) for any \( s \) such that \( \mathcal{H}^s_d(K) < \infty \). In particular, this is the case when \( s \) is the similarity dimension (see [2]) of \( K \). Hence, when \( K \subsetneq \overline{O} \setminus O \), in general we have

\[ \dim_H(K) < \text{similarity dimension of } K. \]

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Although Proposition 1.1 in [6] is not used to get the main results, it must be noted that as proved in [6] the topology of \((\Omega, D)\) is always finer than that of \((\Omega, d)\) but that the topological equivalence, there stated, may fail on the boundary set \(B \cap \mathcal{N}\). However, for any dimension \(s\) for which \(H_d^s(\mathcal{N}) < \infty\), applying again the remarks above, we get the topological equivalence of \(d\) and \(D\) modulo a subset of zero \(H_s^\beta\)-measure and thus of zero \(H_d^s\)-measure. In particular, when \(H_d^s(\mathcal{N}) < \infty\), and this is the case for net fractals satisfying the hypothesis of Theorems 2.2 and 3.1, the space \((\mathcal{N}, d)\) is the disjoint union of an ultrametric subspace and a subset of zero \(H_d^s\)-measure. In particular,

\[\mathcal{N}\]

has the same Hausdorff measure of an

ultametric, topologically zero-dimensional, totally disconnected subspace,

showing that from a measure-theoretic point of view net fractals satisfying the hypothesis of Theorem 3.1 are always ultrametric net fractals.

In the proof of Theorem 3.1 we employed a characterization of Larman’s finite dimensionality due to Rogers [5, p. 104 l.1 and p. 122, Theorem 57 condition (b)], taking for granted that if \(H^n(A) = 0\) for some positive integer \(n\), then \(A\) is finite dimensional in the sense of Larman [4].

While the converse is always true (see [4, corollary to Theorem 4]), this implication is in general false. In fact, for any Hausdorff function \(h(t)\), any set \(A \subseteq \Omega\) contains a countable subset \(A'\) with \(h(A) = h(A')\), in Larman’s notation. But \(A\) and hence \(A'\) may not be finite dimensional.

Let \(\mathcal{N}' : = \{x_i|n\}_{|i|n}\), where \(\{x_i|n\}_{|i|n}\) is the family of point centers of the open balls involved in condition (3) in [6], and call it the expanded net fractal associated with \(\mathcal{N}\).

We have \(\mathcal{N}' = \mathcal{N} \cup \{x_i|n\}_{|i|n}\). In fact, clearly \(\mathcal{N} \subseteq \mathcal{N}'\) and if \(x\) is an accumulation point of \(\{x_i|n\}_{|i|n}\), we can find a sequence \((x_i|n(k))_k\) with \(\lim_{k \to \infty} x_i|n(k) = x\). Since the coordinates \(i_j\) of the curtailed indexes \(i|n(k)\) can assume only a finite number of values, by a standard diagonal argument, we can determine an index, say \(j\), such that \((x_{\tilde{j}|n(k)})_k\) is a subsequence of \((x_i|n(k))_k\) and \(j|n(l + 1)\) is an extension of \(j|n(l)\). It follows that \(x = \lim_{i \to \infty} x_{i|n(l)} = \bigcap_{n=1}^{\infty} A_{i|n} \in \mathcal{N}\) and thus the claim also follows. Moreover, an analogous argument shows that \(\mathcal{N}'\) is sequentially compact and thus compact. Further \(H^s(\mathcal{N}') = H^s(\mathcal{N})\) since \(\{x_i|n\}_{|i|n}\) is countable.

The proof of Theorem 1.3 remains essentially the same if we can use the property

the expanded net fractal \(\mathcal{N}'\) is a \(\beta\)-space.

In general complete metric spaces, in order to guarantee that \(\mathcal{N}'\) is a \(\beta\)-space, we need a further condition concerning the relative position of the points \(\{x_i|n\}_{|i|n}\). However, we are not concerned here with a suitable modification of the basic requirements (1), (2), and (3) in [6] for a net fractal. A complete analysis will appear elsewhere. But we indicate a class of metric spaces, significant from a geometric point of view, in which the above property is automatically satisfied. It is the class of \(locally\ finite-dimensional\ metric spaces\), i.e., the spaces in which every point admits a neighbourhood which is finite dimensional in the sense of Larman [4]. In fact, in these spaces, as we can see using Theorems 11 and 12 in [4], any compact subset is a \(\beta\)-space.
Among the spaces included in this large class, we find the Euclidean spaces and the Riemannian manifolds of class 2.

On page 397 replace lines 11–15 by:

If \( U \cap A_{i_{j_{n(i)}}} \neq \varnothing \), then we can find a ball \( B(x, \, 2\rho) \) in \( \mathcal{N}' \) such that \( U \) and \( A_{i_{j_{n(i)}}} \cap \mathcal{N}' \) are contained in it. Since \( \mathcal{N}' \) is a \( \beta \)-space (for instance with triple \(( M, \delta, \alpha )\)), it follows that at most \( M^q \) disjoint balls of radius \( \rho \alpha^k \) intersect \( B(x, \, 2\rho) \) where \( q \) satisfies \( (2\alpha)^q \leq \alpha^{k+1} < (2\alpha)^q \). Since \( \alpha^k \rho \leq \lambda h \rho \), at most \( M^q \) balls of radius \( \lambda h \rho \) can meet \( B(x, \, 2\rho) \), hence at most \( M^q \) of the sets \( \{ A_{i_1, \ldots, i_{n(i)}} \} \) can meet \( U \).

On page 397 in lines 18 and 20 replace \( M^k \) by \( M^q \).

On page 399 in line 35, replace ‘\( O \) to be bounded’ by ‘\( O \) to be bounded and regular open’.

References


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