We investigate a function which will be used to evaluate the linear recursion

\[ x_i = a_{i,0} + \sum_{j=1}^{i-1} a_{i,j}x_j \quad \text{for } i = 1, 2, 3, \ldots, n \]

where the \( a_{ij} \)'s are arbitrary numbers.

We can express this in matrix notation as \( X = C + AX \) where

\[
C = \begin{bmatrix}
    a_{10} \\
    . \\
    . \\
    . \\
    . \\
    a_{n0}
\end{bmatrix}, \quad
A = \begin{bmatrix}
    0 & 0 & 0 & \cdots & 0 \\
    a_{21} & 0 & 0 & \cdots & 0 \\
    a_{31} & a_{32} & 0 & \cdots & 0 \\
    . & . & . & \cdots & . \\
    . & . & . & \cdots & . \\
    a_{n1} & a_{n2} & a_{n3} & \cdots & 0
\end{bmatrix}, \quad
X = \begin{bmatrix}
    x_1 \\
    . \\
    . \\
    . \\
    . \\
    x_n
\end{bmatrix}
\]

\( A \) is referred to as the coefficient matrix, \( C \) as the constant vector, and \( X \) as the solution vector.

**Definition.** For \( 0 < j < 2^n \), \( 1 \leq i \leq 2^n \), we define a sequence of functions \( f_0(i, j) \), \( f_1(i, j) \), \( f_2(i, j) \), \ldots, \( f_n(i, j) \) such that

\[
f_{r+1}(i, j) = \begin{cases} 
    f_r(i, j) + \sum_{k=j+2^r-b}^{j+2^{r+1}-b-1} f_r(i, k) f_r(k, j) & \text{if } j \equiv b \pmod{2^{r+1}}, \\
    f_r(i, j) & \text{otherwise}
\end{cases}
\]

for \( 0 \leq r < n \), with

\[ f_0(i, j) = a_{i,j} \quad \text{and} \quad a_{i,j} = 0 \quad \text{for } i \leq j. \]

**Remark.** \( a_{i,j} = 0 \) for \( i \leq j \) implies \( f_r(i, j) = 0 \) for \( i \leq j \).

By repeatedly applying this recursive definition we can express any \( f_{r+1}(i, j) \) in terms of \( f_r, f_{r-1}, \ldots \), and finally \( f_0 \), and thus express \( f_{r+1}(i, j) \) as a function of \( a \)'s only.
Examples.

\[ f_1(4, 0) = f_0(4, 0) + f_0(4, 1)f_0(1, 0) = a_{4, 0} + a_{4, 1}a_{1, 0}, \]
\[ f_1(5, 1) = f_0(5, 1) = a_{5, 1}, \]
\[ f_2(3, 0) = f_1(3, 0) + f_1(3, 2)f_1(2, 0) \]
\[ = f_0(3, 0) + f_0(3, 1)f_0(1, 0) + f_0(3, 2)[f_0(2, 0) + f_0(2, 1)f_0(1, 0)] \]
\[ = a_{3, 0} + a_{3, 1}a_{1, 0} + a_{3, 2}(a_{2, 0} + a_{2, 1}a_{1, 0}). \]

Theorem. Let \( j \equiv b \mod 2^r \), \( 0 < b < 2^r \); then

\[ f_r(i, j) = a_{i, j} + \sum a_{i, j(1)}a_{j(1), j(2)} \cdots a_{j(u), j} \]

where the sum is over all \( u \)-tuples \((j(1), \ldots, j(u))\) satisfying \( j < j(1) < \cdots < j(u) < \min\{j + 2^r - b, i\}\), where \( 0 < u < \min\{2^r - b, i - j\} \).

Proof. Our proof is by induction on \( r \). We observe that

\[ f_0(i, j) = a_{i, j}, \]
\[ f_1(i, j) = \begin{cases} a_{i, j} + a_{i, j+1}a_{j+1, j} & \text{if } j \equiv 0 \mod 2, \\ a_{i, j} & \text{otherwise.} \end{cases} \]

So the result is verified for \( r = 0 \) and \( r = 1 \).

Assume the result is valid for \( r = s \). Now using this, we will deduce the corresponding result for \( s + 1 \):

\[ f_{s+1}(i, j) = a_{i, j} + \sum a_{i, j(1)}a_{j(1), j(2)} \cdots a_{j(u), j} \]

where the sum is over all \( u \)-tuples \((j(1), \ldots, j(u))\) satisfying \( j < j(1) < \cdots < j(u) < \min\{j + 2^{s+1} - b, i\}\), where \( 0 < u < \min\{2^{s+1} - b, i - j\} \) and \( j \equiv b \mod 2^{s+1} \), \( 0 < b < 2^{s+1} \).

We consider three cases.

Case 1. \( j \equiv b \mod 2^{s+1} \), \( 2^s \leq b < 2^{s+1} \). In this case \( f_{s+1}(i, j) = f_s(i, j) \) by definition, and by the induction hypothesis

\[ f_{s+1}(i, j) = a_{i, j} + \sum a_{i, j(1)}a_{j(1), j(2)} \cdots a_{j(u), j}, \]

where \( 0 < u < \min\{2^s - b', i - j\} \) and \( j \equiv b' \mod 2^s \), \( 0 < b' < 2^s \).

But \( j \equiv b \mod 2^{s+1} \), \( 2^s \leq b < 2^{s+1} \), implies \( j \equiv b - 2^s \mod 2^s \), \( 0 \leq b < 2^s \), and \( b' = b - 2^s \). Therefore

\[ f_{s+1}(i, j) = a_{i, j} + \sum a_{i, j(1)}a_{j(1), j(2)} \cdots a_{j(u), j}, \]

where \( 0 < u < \min\{2^{s+1} - b, i - j\} \) and \( j \equiv b \mod 2^{s+1} \), \( 2^s \leq b < 2^{s+1} \).

Case 2. \( j \equiv b \mod 2^{s+1} \), \( 0 \leq b < 2^s \), and \( i < j + 2^s - b \). It follows by definition that \( f_{s+1}(i, j) = f_i(i, j) \) and by the induction hypothesis

\[ f_{s+1}(i, j) = a_{i, j} + \sum a_{i, j(1)}a_{j(1), j(2)} \cdots a_{j(u), j}, \]

where \( 0 < u < \min\{2^s - b', i - j\} \) and \( j \equiv b' \mod 2^s \), \( 0 < b' < 2^s \). Since \( j \equiv b \mod 2^{s+1} \), \( 0 \leq b < 2^s \), implies \( j \equiv b \mod 2^s \), \( 0 \leq b < 2^s \), we have
b' = b. Then \( \min\{j + 2^s - b, i\} = i = \min\{j + 2^{s+1} - b, i\} \), since \( i \leq j + 2^s - b \). Therefore

\[
fs_{+1}(i, j) = a_{i, j} + \sum a_{i, j(1)}a_{j(1), j(2)}\cdots a_{j(u), j},
\]

\( j < j(u) < \cdots < j(1) < \min\{j + 2^{s+1} - b, i\} \),

where \( 0 < u < \min\{2^{s+1} - b, i - j\} \) and \( j \equiv b \pmod{2^{s+1}} \), \( 0 \leq b < 2^s \), and \( i \leq j + 2^s - b \).

**Case 3.** \( j \equiv b \pmod{2^{s+1}} \), \( 0 \leq b < 2^s \), and \( i > j + 2^s - b \). By definition

\[
f_{s+1}(i, j) = f_{s}(i, j) + \sum_{k = j + 2^s - b}^{j + 2^{s+1} - b - 1} \sum_{\min\{j + 2^s + x - b, i\} - 1} f_{s}(i, k)f_{s}(k, j).
\]

From the induction hypothesis

\[
f_{s}(i, k) = a_{i, k} + \sum a_{i, k(1)}a_{k(1), k(2)}\cdots a_{k(u), k},
\]

\( k < k(u) < \cdots < k(1) < \min\{k + 2^s - c, i\} \),

where \( 0 < u < \min\{2^s - c, i - k\} \) and \( k \equiv c \pmod{2^s} \), \( 0 \leq c < 2^s \). When \( j + 2^s - b \leq k < j + 2^{s+1} - b \), \( k = j + 2^s - b + c \) and \( k + 2^s - c = j + 2^{s+1} - b \). Therefore, for all \( k \) such that \( j + 2^s - b \leq k < j + 2^{s+1} - b \) we have

(i)

\[
f_{s}(i, k) = a_{i, k} + \sum a_{i, k(1)}a_{k(1), k(2)}\cdots a_{k(u), k},
\]

\( k < k(u) < \cdots < k(1) < \min\{j + 2^{s+1} - b, i\} \),

where \( 0 < u < \min\{j + 2^{s+1} - b, i\} - k \) and \( j \equiv b \pmod{2^{s+1}} \), \( 0 \leq b < 2^s \).

Also, by the induction hypothesis

\[
f_{s}(k, j) = a_{k, j} + \sum a_{k, j(1)}a_{j(1), j(2)}\cdots a_{j(u), j},
\]

\( j < j(v) < \cdots < j(1) < \min\{j + 2^s - b', k\} \),

where \( 0 < v < \min\{2^s - b', k - j\} \) and \( j \equiv b' \pmod{2^s} \), \( 0 \leq b' < 2^s \). But since \( j \equiv b \pmod{2^{s+1}} \), \( 0 \leq b < 2^s \), it follows that \( b' = b \), and when \( j + 2^s - b \leq k < j + 2^{s+1} - b \) we have \( \min\{j + 2^s - b, k\} = j + 2^s - b \). Therefore, for all \( k \) such that \( j + 2^s - b \leq k < j + 2^{s+1} - b \)

(ii)

\[
f_{s}(k, j) = a_{k, j} + \sum a_{k, j(1)}a_{j(1), j(2)}\cdots a_{j(u), j},
\]

\( j < j(v) < \cdots < j(1) < j + 2^s - b \),

where \( 0 < v < 2^s - b \) and \( j \equiv b \pmod{2^{s+1}} \), \( 0 \leq b < 2^s \).
It follows from (i) and (ii) that

$$\min\{j + 2^{s+1} - b, i\} - 1 \sum_{k=j+2^s-b}^{\min\{j + 2^{s+1} - b, i\} - 1} f_s(i, k) f_s(k, j)$$

$$= \sum_{k=j+2^s-b}^{\min\{j + 2^{s+1} - b, i\} - 1} \left( a_i, k, a_j, j + \sum_{j(v)} a_i, k, a_j, j(1) \cdots a_j(v), j \right)$$

$$+ \sum_{k(u)} a_i, k(1) \cdots a_k(u), k a_k, j$$

$$+ \sum_{k(v)} a_i, k(1) \cdots a_k(u), k a_k, j(1) \cdots a_j(v), j$$

where \( k < k(u) < \cdots < k(1) < \min\{j + 2^{s+1} - b, i\} \),

\( j < j(v) < \cdots < j(1) < j + 2^s - b \),

\( 0 < u < \min\{j + 2^{s+1} - b, i\} - k \),

\( 0 < v < 2^s - b \), and \( j \equiv b \) (mod \( 2^{s+1} \)), \( 0 \leq b < 2^s \).

By removing the outer summation symbol and making a change of variables, we obtain

$$\min\{j + 2^{s+1} - b, i\} - 1 \sum_{k=j+2^s-b}^{\min\{j + 2^{s+1} - b, i\} - 1} f_s(i, k) f_s(k, j)$$

$$= \sum_{C(1)} a_i, j(1) a_j(1), j + \sum_{C(2)} a_i, j(1) a_j(1), j(2) \cdots a_j(v+1), j$$

$$+ \sum_{C(3)} a_i, j(1) \cdots a_j(u), j(u+1) a_j(u+1), j$$

$$+ \sum_{C(4)} a_i, j(1) \cdots a_j(u), j(u+1) a_j(u+1), j(u+2) \cdots a_j(u+v+1), j$$

where \( C(1), C(2), C(3), C(4) \) are the summation conditions:

\( C(1) \): \( j + 2^s - b \leq j(1) < \min\{j + 2^{s+1} - b, i\} \) where \( j \equiv b \) (mod \( 2^{s+1} \)), \( 0 \leq b < 2^s \),

\( C(2) \): \( j < j(v + 1) < \cdots < j(2) < j + 2^s - b \leq \min\{j + 2^{s+1} - b, i\} \) where

\( 0 < v < 2^s - b \) and \( j \equiv b \) (mod \( 2^{s+1} \)), \( 0 \leq b < 2^s \),

\( C(3) \): \( j + 2^s - b \leq j(u + 1) < j(u) < \cdots < j(1) < \min\{j + 2^{s+1} - b, i\} \) where

\( 0 < u < \min\{j + 2^{s+1} - b, i\} - j(u + 1) \) and \( j \equiv b \) (mod \( 2^{s+1} \)), \( 0 \leq b < 2^s \),

\( C(4) \): \( j < j(u + v + 1) < \cdots < j(u + 2) < j + 2^s - b \leq j(u + 1) < j(u) < \cdots < j(1) < \min\{j + 2^{s+1} - b, i\} \) where

\( 0 < u < \min\{j + 2^{s+1} - b, i\} - j(u + 1) \), \( 0 < v < 2^s - b \), and \( j \equiv b \) (mod \( 2^{s+1} \)), \( 0 \leq b < 2^s \).

Remarks.
1. \( u + v + 1 < \min\{2^{s+1} - b, i - j\} \).
2. \( C(1), C(2), C(3), C(4) \) describe a partition of the condition

\( j + 2^s - b \leq j(1) < \min\{j + 2^{s+1} - b, i\} \), \( j < j(w) < \cdots < j(2) < j(1) \).
where \( 0 < w < \min\{2^{s+1} - b, i - j\} \) and \( j \equiv b \pmod{2^{s+1}} \), \( 0 \leq b < 2^s \).

We now conclude

\[
\min\{j + 2^{s+1} - b, i\} - 1
\]

\[
\sum_{k=j+2^i-b}^{j} f_s(i, k) f_s(k, j) = \sum a_{i,j(1)} \cdots a_{j(w),j} ,
\]

\( j + 2^s - b \leq j(1) < \min\{j + 2^{s+1} - b, i\}, \)

\( j < j(w) < \cdots < j(2) < j(1), \)

where \( 0 < w < \min\{2^{s+1} - b, i - j\} \) and \( j \equiv b \pmod{2^{s+1}} \), \( 0 \leq b < 2^s \). It follows from the induction hypothesis that

\[
f_s(i, j) = a_{i,j} + \sum_{S(1)} a_{i,j(1)} \cdots a_{j(x),j} ,
\]

\( j < j(x) < \cdots < j(1) < \min\{j + 2^s - b', i\}, \)

\( 0 < x < \min\{2^s - b', i - j\} \) and \( j \equiv b' \pmod{2^s} \), \( 0 \leq b' < 2^s \).

We have \( j \equiv b \pmod{2^{s+1}} \), \( 0 \leq b < 2^s \), therefore \( b' = b \), and since \( i > j + 2^s - b \) the \( \min\{j + 2^s - b, i\} = j + 2^s - b \).

From (iii) and (iv) we conclude that when \( j \equiv b \pmod{2^{s+1}} \), \( 0 \leq b < 2^s \), and \( i > j + 2^s - b \), then

\[
f_{s+1}(i, j) = a_{i,j} + \sum_{S(1)} a_{i,j(1)} \cdots a_{j(y),j} \]

where \( S(1) \) and \( S(2) \) denote the summation conditions

\( j < j(x) < \cdots < j(1) < j + 2^s - b, \quad 0 < x < 2^s - b, \)

and \( j + 2^s - b \leq j(1) < \min\{j + 2^{s+1} - b, i\}, \)

\( j < j(w) < \cdots < j(2) < j(1), \quad 0 < w < \min\{2^{s+1} - b, i - j\}, \)

respectively. Hence we have

\[
f_{s+1}(i, j) = a_{i,j} + \sum_{S(1)} a_{i,j(1)} a_{j(1),j(2)} \cdots a_{j(y),j} ,
\]

\( j < j(y) < \cdots < j(1) < \min\{j + 2^{s+1} - b, i\}, \)

where \( 0 < y < \min\{2^{s+1} - b, i - j\} \) and \( j \equiv b \pmod{2^{s+1}} \), \( 0 \leq b < 2^s \), and \( i > j + 2^s - b \).

Together, Cases 1, 2, and 3 show that the theorem holds for \( r = s + 1 \), and thus for all nonnegative integers less than or equal to \( n \). \( \square \)

**Corollary.** \( f_n(i, 0) = a_{i,0} + \sum_{j=1}^{i-1} a_{i,j} x_j = x_i \) for \( 1 \leq i \leq 2^n \).

**Proof.** Our proof is by induction on \( i \). The corollary is true for \( i = 1 \) and \( i = 2 \), since \( f_n(1, 0) = a_{1,0} = x_1 \) for \( n \geq 0 \) and \( f_n(2, 0) = a_{2,0} + a_{2,1} a_{1,0} = x_2 \) for \( n \geq 1 \). As an induction hypothesis, assume it holds for all \( i \leq k < 2^n \). Thus, by hypothesis we have

\[
f_n(i, 0) = a_{i,0} + \sum_{j=1}^{i-1} a_{i,j} x_j = x_i \quad \text{for} \quad 1 \leq i \leq k. \]

Now we prove the lemma for \( i = k + 1 \). We know by definition that

\[
x_{k+1} = a_{k+1,0} + \sum_{j=1}^{k} a_{k+1,j} x_j ,
\]
and by the induction hypothesis this can be written as

$$x_{k+1} = a_{k+1,0} + \sum_{j=1}^{k} a_{k+1,j} f_n(j, 0).$$

It follows from our theorem that

$$x_{k+1} = a_{k+1,0} + \sum_{j=1}^{k} a_{k+1,j} \left( a_j, 0 + \sum a_{j, j(1)} a_{j(1), j(2)} \cdots a_{j(u), 0} \right),$$

$$0 < j(u) < \cdots < j(1) < \min\{2^n, j\}, \quad 0 < u < \min\{2^n, j\},$$

and we obtain

$$x_{k+1} = a_{k+1,0} + \sum_{1 \leq j(1) \leq k} a_{k+1, j(1)} a_{j(1), 0} + \sum a_{k+1, j(1)} \cdots a_{j(u+1), 0},$$

$$0 < j(u+1) < \cdots < j(2) < j(1) \leq k, \quad 0 < u < j(1).$$

Therefore,

$$x_{k+1} = a_{k+1,0} + \sum a_{k+1+j(1)} \cdots a_{j(v), 0},$$

$$0 < j(v) < \cdots < j(1) \leq k, \quad 0 < v \leq k,$$

which can be written as

$$x_{k+1} = a_{k+1,0} + \sum a_{k+1+j(1)} \cdots a_{j(v), 0},$$

$$0 < j(v) < \cdots < j(1) \leq \min\{2^n, k+1\}, \quad 0 < v \leq \min\{2^n, k+1\},$$

and it follows by our theorem that

$$x_{k+1} = f_n(k+1, 0). \quad \Box$$

Based on the result that $f_n(i, 0) = x_i$ for $1 \leq i \leq 2^n$ Chen and Kuck [CK] give a parallel algorithm for evaluating $x_i$, $1 \leq i \leq 2^n$. Time and processor bounds for solving the linear recurrence system are then obtained. Sameh and Brent [SB] later presented an alternate derivation of this algorithm.

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