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A NOTE ON MINIMAL PRIME IDEALS

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ABSTRACT. Let R be a commutative ring and I an ideal of R. We show that if all the prime ideals minimal over I are finitely generated, then there are only finitely many prime ideals minimal over I.

Let R be a commutative ring with identity, and let $I \neq R$ be an ideal of R. It is well known that if R is Noetherian then there are only finitely many prime ideals minimal over I. In fact, we only need R to satisfy the ascending chain condition on radical ideals [1, Theorem 88]. Of course, for R Noetherian every prime ideal minimal over I is finitely generated. The purpose of this note is to show that for any commutative ring if every prime ideal minimal over I is finitely generated, then there are only finitely many prime ideals minimal over I. It is interesting to note that if we take R to be the ring of all sequences from $Z/2Z$ that are eventually constant, with pointwise addition and multiplication, then R is a zero-dimensional Boolean ring with minimal prime ideals $M_i = \{\{a_n\} \in R | a_i = 0\}$ and $M_\infty = \{\{a_n\} \in R | a_n = 0$ for large $n\}$ and each M_i is principal but M_∞ is not finitely generated. Thus while R has infinitely many minimal primes ideals, only one is not finitely generated.

Theorem. *Let R be a commutative ring with identity, and let $I \neq R$ be an ideal of R. If every prime ideal minimal over I is finitely generated, then there are only finitely many prime ideals minimal over I.*

Proof. Let $S = \{P_1 \cdots P_n |$ each P_i is a prime ideal minimal over $I\}$. If for some $C = P_1 \cdots P_n \in S$ we have $C \subseteq I$, then any prime ideal P minimal over I contains some P_i, so $\{P_1, \dots, P_n\}$ is the set of minimal prime ideals of I. Hence we may assume that $C \not\subseteq I$ for each $C \in S$. Consider the set $T = \{J | J$ is an ideal of R with $J \supseteq I$ and $C \not\subseteq J$ for each $C \in S\}$ partially ordered by $\subseteq$. Since each element of S is finitely generated, T is inductive and hence by Zorn's Lemma has a maximal element Q. It is easily seen that Q is prime. But then since $Q \supseteq I$, Q contains a prime ideal P minimal over I [1, Theorem 10]. Thus $P \in S$, a contradiction.

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References

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