THE INVERSE CONDUCTIVITY PROBLEM WITH ONE MEASUREMENT: UNIQUENESS FOR CONVEX POLYHEDRA

BARTOLOMÉ BARCELÓ, EUGENE FABES, AND JIN KEUN SEO

(Communicated by Barbara L. Keyfitz)

Abstract. Let $\Omega$ denote a smooth domain in $\mathbb{R}^n$ containing the closure of a convex polyhedron $D$. Set $\chi_D$ equal to the characteristic function of $D$. We find a flux $g$ so that if $u$ is the nonconstant solution of $\text{div} ((1 + \chi_D) \nabla u) = 0$ in $\Omega$ with $\frac{\partial u}{\partial n} = g$ on $\partial \Omega$, then $D$ is uniquely determined by the Cauchy data $g$ and $f = u|_{\partial \Omega}$.

Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a connected boundary and $D$ a subdomain in $\Omega$. Assume both $\Omega$ and $D$ are conductors of electricity. We consider the following question: Can we set up a magnetic field $E$ surrounding $\Omega$ with a known flux $g$ across $\partial \Omega$ so that calculating the potential of the field on $\partial \Omega$ will determine $D$?

Writing $E = \nabla u$ we have

$$L_x u = \text{div} (\gamma(x) \nabla u) = 0 \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial \Omega,$$

where $\gamma(x)$ is the conductivity and $\frac{\partial u}{\partial n}$ denotes the normal derivative of $u$ on $\partial \Omega$. The question now becomes: Can we choose $g$ so that $g$ and $f = u|_{\partial \Omega}$ uniquely determine $D$?

For the sake of definiteness we take $\gamma(x) = 1 + \chi_D(x)$, where $\chi_D$ denotes the characteristic function of $D$. In this case, Friedman and Isakov [1] proved that there is a flux $g$ so that $g$ and $f$ uniquely determine $D$ if $D$ is assumed to be a convex polyhedron situated away from the boundary of $\Omega$, that is, if

$$\text{diam}(D) < \text{dist}(D, \partial \Omega).$$

In this paper we are able to remove this extra condition. Specifically, we prove the following uniqueness result: There exists a function $g$ defined on
\[ \partial \Omega \] such that if \( D_1, D_2 \) are convex polyhedra compactly contained in \( \Omega \) and \( u_i, i = 1, 2, \) is the solution of
\[
\text{div} ((1 + \chi_{D_i}(x)) \nabla u_i(x)) = 0 \quad \text{in} \quad \Omega,
\]
\[
\int_{\Omega} u_i \, dx = 0 \quad \text{and} \quad \frac{\partial u_i}{\partial n} \bigg|_{\partial \Omega} = g,
\]
then if \( u_1|_{\partial \Omega} = u_2|_{\partial \Omega} \), we conclude \( D_1 = D_2 \). We prove this result in §3 (Theorem 1) for the case of convex polygons. The proof for convex polyhedra in \( R^n \) follows from Lemma 3 using similar arguments.

2. Preliminaries

We recall that by definition a convex polyhedron \( P \) in \( R^n \) is a finite intersection of half spaces. That is, if
\[
H_i = \{ x \in R^n : (x - x_i) \cdot n_i > 0 \}
\]
for some points \( x_1, x_2, \ldots, x_k \in R^n \) and unit vectors \( n_1, n_2, \ldots, n_k \in R^n \), then
\[
P = H_1 \cap H_2 \cap \cdots \cap H_k.
\]
An edge \( \gamma \) of the polyhedron \( P \) is the intersection of two faces, i.e., the intersection of two hyperplanes used in the definitions of \( P \). A polyhedron \( D \) in \( R^n \) is an open connected set that is the union of convex polyhedrons.

Given a bounded \( C^2 \) domain \( \Omega \) in \( R^n \) and a polyhedron \( D \) compactly supported in \( \Omega \), let us consider the Neumann problem
\[
\begin{cases}
Lu = \text{div} ((1 + \chi_D(x)) \nabla u) = 0 & \text{in} \quad \Omega,
\end{cases}
\]
\[
\frac{\partial u}{\partial n} = g & \text{on} \quad \partial \Omega,
\]
where \( n \) is the unit normal to the boundary and \( u \) is normalized by \( \int_{\Omega} u = 0 \).

By a weak solution \( u \) to \( Lu = 0 \) we mean a function \( u \in L^2(\Omega) \) whose partial derivatives \( u_{x_i} \), in the distributional sense, belong to \( L^2(\Omega) \) and such that
\[
\sum_{i=1}^{n} \int_{\Omega} (1 + \chi_D) \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_i} \, dx = 0
\]
for all \( \phi \in C_0^\infty(\Omega) \), the class of \( C^\infty \) functions with compact support in \( \Omega \).

It is well known that if \( g \in L^\infty(\partial \Omega) \), then there exists a unique solution \( u \) to (1) such that \( u \in C^\alpha(\Omega) \) for some \( \alpha > 0 \).

If we set
\[
u^e = u|_{\Omega \setminus D} \quad \text{and} \quad u^i = u|_D,
\]
then \( \Delta u^e = 0 \) in \( \Omega \setminus D \), \( \Delta u^i = 0 \) in \( D \), and \( u^e = u^i \) on \( \partial D \).

Also, by integrating by parts in (1) we get
\[
\frac{\partial u^e}{\partial n} = 2 \frac{\partial u^i}{\partial n}
\]
on the smooth part of \( \partial D \).

3. The main theorem

Let \( \Omega \subset R^n \) be a bounded convex \( C^2 \) domain in \( R^n \) with a connected boundary. Let \( D_1, D_2 \) be two polyhedra compactly supported in \( \Omega \). For
i = 1, 2 consider

\[ \begin{cases} L_{D_i}u_i = \text{div} \ ((1 + \chi_{D_i}(x))\nabla u_i) = 0 \quad \text{in } \Omega, \\
\frac{\partial u_i}{\partial n} = g \quad \text{on } \partial \Omega, \end{cases} \tag{4} \]

where we also take \( \int_{\Omega} u_i \, dx = 0. \)

Let us fix a \( L^\infty(\partial \Omega) \) function \( g \) such that there is no harmonic function \( u \) on any neighborhood \( V \) of any point of \( \partial \Omega \) with \( g = \nabla u \cdot n \) on \( V \cap \partial \Omega \). For example, let us take a dense sequence of points \( \{p_k\} \) on the surface boundary \( \partial \Omega \) and consider the function

\[ g(x) = \sum_{k=1}^{\infty} 2^{-k} |x - p_k| - c, \tag{5} \]

where \( x \in \mathbb{R}^n \) and \( c \) is chosen so that \( \int_{\partial D} g \, d\sigma = 0. \) The function \( g \) is Lipschitz but not \( C^1 \) on any surface ball \( B(p) \cap \partial \Omega. \) If \( u \) is harmonic in \( B(p), \) then \( \nabla u \cdot n \) is \( C^1(B(p)) \cap \partial \Omega, \) so we cannot have \( \nabla u \cdot n = g \) there.

**Theorem 1.** Let \( D_1, D_2 \) be convex polygons in \( \mathbb{R}^2. \) Assume \( u_i, i = 1, 2, \) are the solutions to the Neumann problem (4) with \( g \) as in (5). If \( u_1 = u_2 \) on \( \partial \Omega, \) then \( D_1 = D_2. \)

**Proof.** If \( D_1 \) and \( D_2 \) have the same corners, there is nothing to be proved. Let us assume that \( P \) is a corner of \( D_1 \) but \( P \) is not a corner of \( D_2. \)

We can also assume, for simplicity, that \( P \) is the origin of \( \mathbb{R}^2 \) and, in polar coordinates, if \( B_{n_0} \) is a small ball centered at the origin,

\[ B_{n_0} \cap D_1 = \{(r, \theta) : 0 < r < r_0, 0 < \theta < \theta_0\} \]

and

\[ B_{n_0} \cap D_2 = \emptyset. \]

Note that \( 0 < \theta_0 < \pi \) because of the convexity of \( D_1. \)

Since \( u_1 = u_2 \) on \( \partial \Omega \) and also \( \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = g \) on \( \partial \Omega, \) by the unique continuation property for harmonic functions it follows that \( u_1 = u_2 \) on the connected component of \( \Omega \setminus \{D_1 \cup D_2\} \) near the boundary \( \partial \Omega. \) Since \( u_2 \) is harmonic on \( B_{n_0}, \) \( u_1|_{B_{n_0} \setminus D_1} \) has a harmonic extension to the whole ball \( B_{n_0}. \) (In fact, this harmonic extension is \( u_2 \) itself.) Let us call \( u_1^{\text{ext}} \) this harmonic extension.

From Lemma 3 below, in polar coordinates, we have

\[ u_1^{\text{ext}}(r, \theta) = u_1^{\text{ext}}(r, \theta \pm 2\theta_0) \tag{6} \]

for all \( (r, \theta) \in B_{n_0} \) for some \( 0 < \epsilon < r_0. \)

Let us represent by \( \tilde{u}_1 \) the rotation of \( u_1 \) by the angle \( 2\theta_0, \) that is,

\[ \tilde{u}_1(r, \theta) \equiv u_1(r, \theta - 2\theta_0), \]

and let us denote by \( \tilde{\Omega}, \tilde{D}_1 \) the rotated corresponding domains; that is,

\[ \tilde{\Omega} = \{(r, \theta + 2\theta_0) : (r, \theta) \in \Omega\}, \]

\[ \tilde{D}_1 = \{(r, \theta + 2\theta_0) : (r, \theta) \in D_1\}. \]

Since

\[ u_1(r, \theta) = u_1^{\text{ext}}(r, \theta) \quad \text{for } (r, \theta) \in B_{n_0} \setminus D_1, \]

\[ \tilde{u}_1(r, \theta) = u_1^{\text{ext}}(r, \theta - 2\theta_0) \quad \text{for } (r, \theta) \in B_{n_0} \setminus \tilde{D}_1, \]

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
it follows from (6) that
\[ u_1 = \hat{u}_1 \quad \text{in} \quad (B_\epsilon \setminus D_1) \cap (B_\epsilon \setminus \hat{D}_1). \]

Now, by unique continuation, we have \( u_1 = \hat{u}_1 \) on all open connected components near the origin of \((\Omega \cap \hat{\Omega}) \setminus (D_1 \cup \hat{D}_1)\).

Since \( \theta_0 < \pi \), there are \( \theta_1 \) and \( \delta > 0 \) such that if \( \theta_1 < \phi < \theta_1 + \delta \), then the ray
\[ T_\phi = \{(r, \theta) : r > 0, \theta = \phi\} \]
does not intersect \( D_1 \cup \hat{D}_1 \). Let \( Q_\phi \) and \( \hat{Q}_\phi \) be the points on \( \partial \Omega \) and \( \partial \hat{\Omega} \) respectively such that
\[ (0, Q_\phi) \subset \Omega \cap T_\phi \quad \text{and} \quad (0, \hat{Q}_\phi) \subset \hat{\Omega} \cap T_\phi, \]
where \((0, Q_\phi)\) denotes the open line segment joining the origin \(0\) with \(Q_\phi\) and \((0, \hat{Q}_\phi)\) the open line segment joining \(0\) with \(\hat{Q}_\phi\). We will prove
\[ Q_\phi = \hat{Q}_\phi \quad \text{for all} \quad \theta_1 < \phi < \theta_1 + \delta. \]

If not, we may assume \((0, Q_\phi) \subset (0, \hat{Q}_\phi)\). Since \( u_1 \) and \( \hat{u}_1 \) coincide on \((0, Q_\phi)\) and \( \hat{u}_1 \) is harmonic in a neighborhood of \( Q_\phi \), a contradiction follows because \( u_1 \) cannot be harmonically extended across any small neighborhood of the point \( Q_\phi \) because of the bad boundary data \( g \). This proves that if
\[ \mathcal{S} = \bigcup_{\theta_1 < \phi < \theta_1 + \delta} (0, Q_\phi), \]
then
\[ \Omega \cap \mathcal{S} = \hat{\Omega} \cap \mathcal{S} = \mathcal{S}, \quad \bigcup_{\theta_1 < \phi < \theta_1 + \delta} \{Q_\phi\} \subset \partial \Omega \cap \partial \hat{\Omega}. \]

Now we will prove \( \partial \Omega = \partial \hat{\Omega} \). Let \( P_0 \in \bigcup_{\theta_1 < \phi < \theta_1 + \delta} \{Q_\phi\} \), and let \( d = \frac{1}{2} \text{dist}(D_1, \partial \Omega) \). Now we define
\[ s(P_0) = \text{Sup} \{r : B_r(P_0) \text{ has the property \( \forall \)} \}
\[ \text{there exists a ball } B_\epsilon(P) \text{ such that } B_\epsilon(P) \cap \Omega = B_\epsilon(P) \cap \hat{\Omega} \]
and \( B_\epsilon(P) \cap \Omega^c = B_\epsilon(P) \cap \hat{\Omega}^c \} \).

We claim \( s(P_0) = d \). Suppose \( s < d \). We can pick \( P \in \partial B_{s}(P_0) \cap \partial \Omega \). Clearly \( P \in \partial \Omega \cap \partial \hat{\Omega} \).

Recall \( \Omega \) is convex. We can choose a small ball \( B(P) \) such that in local coordinates there exist functions \( \psi \) and \( \hat{\psi} \) such that
\[ B(P) \cap \Omega = \{(x, y) : y < \psi(x)\} \cap B(P) \]
and
\[ B(P) \cap \hat{\Omega} = \{(x, y) : y < \hat{\psi}(x)\} \cap B(P). \]
From the uniqueness of harmonic continuation, it is easy to see that
\[ u_1 = \hat{u}_1 \quad \text{in} \quad B_{s}(P_0) \cap \Omega \cap \hat{\Omega}, \]
so
\[ u_1 = \hat{u}_1 \quad \text{in} \quad B(P) \cap \Omega \cap \hat{\Omega}. \]
Let $P = (x_0, \psi(x_0))$. Then there is a small $\epsilon_1 > 0$ such that

$$(x, \psi(x)), (x, \psi(x)) \in B(P) \quad \text{whenever } |x - x_0| < \epsilon_1.$$ 

We will show $B_{\epsilon_1}(P) \cap \Omega = B_{\epsilon_1}(P) \cap \tilde{\Omega}$. It suffices to show

$$\psi(x) = \tilde{\psi}(x) \quad \text{for } |x - x_0| < \epsilon_1.$$ 

If not, there is an $x$ with $|x - x_0| < \epsilon_1$ such that $\psi(x) < \tilde{\psi}(x)$ or $\tilde{\psi}(x) < \psi(x)$. Suppose $\psi(x) < \tilde{\psi}(x)$. Then $(x, \psi(x)) \in \tilde{\Omega} \cap B(P)$, and therefore there is a small ball $B_{\epsilon_2}(Q)$ where $Q = (x, \psi(x))$ such that $B_{\epsilon_2}(Q) \subset B(P) \cap \tilde{\Omega}$. Since

$$u_1 = \tilde{u}_1 \quad \text{in } B_{\epsilon_2}(Q) \cap \Omega \subset B(P) \cap \Omega \cap \tilde{\Omega}$$

and $\tilde{u}_1$ is harmonic in $B_{\epsilon_2}(Q)$, $u_1$ has a harmonic extension to $B_{\epsilon_2}(Q)$, and this is not possible because of the bad boundary data $g$. This proves $s(P_0) = \partial Q$. Next we pick $P_1 \in \partial \Omega \cap \partial B_{d/2}(P_0)$. Then the same arguments as above show that $s(P_1) = d$, and by repeating this process we conclude $\partial \Omega = \partial \tilde{\Omega}$ or $\Omega = \tilde{\Omega}$. Hence $u_1$ has a harmonic extension to

$$\left( \Omega \setminus \overline{D_1} \right) \cup \left( \Omega \setminus \overline{D_1} \right).$$

We can repeat the same arguments as before for the $k$-rotation of $u_1$

$$\tilde{u}_1^k(r, \theta) = u_1(r, \theta - 2k\theta_0)$$

for $k = 0, 1, 2, \ldots$ in such a way that any consecutive two rotations $\tilde{u}_1^k$ and $\tilde{u}_1^{k+1}$ play the role of $u_1$ and $\tilde{u}_1$ respectively, concluding finally that $u_1$ has a harmonic extension to

$$\left( \Omega \setminus \overline{D_1} \right) \cup \left( \Omega \setminus \overline{D_1} \right) \cup \cdots \cup \left( \Omega \setminus \overline{D_1^k} \right).$$

Here $D_1^k = \{(r, \theta + 2k\theta_0) : (r, \theta) \in D_1\}$.

We claim that this last set is $\Omega \setminus \{0\}$ for some finite $k$. Since $2\theta_0 < 2\pi$, there is the positive integer $k_0$ such that

$$2k_0 + 1)\theta_0 < 2k_0\pi \quad \text{and} \quad (2k + 1)\theta_0 \geq 2k\pi \quad \text{for } k = 0, 1, \ldots, k_0 - 1.$$  

We will show

$$\Omega = \left( \Omega \setminus \overline{D_1} \right) \cup \left( \Omega \setminus \overline{D_1} \right) \cup \left( \Omega \setminus \overline{D_1^{k_0}} \right) \cup \{0\}.$$  

In fact, if that is not true, there is an $(r, \theta) \in \Omega$ such that

$$(r, \theta) \in D_1 \cap \overline{D_1} \cap \overline{D_1^{k_0}}.$$  

From (7) we can get

$$0 \leq (2k_0 - 1)\theta_0 - 2(k_0 - 1)\pi < (2k_0 + 1)\theta_0 - 2(k_0 - 1)\pi < 2\pi,$$

where the first inequality comes from the second statement in (7) with $k = k_0 - 1$ and the last inequality comes from the first statement in (7).

Also, we can rewrite the first inequality in (9) as

$$\theta_0 \leq 2k_0\theta_0 - 2(k_0 - 1)\pi.$$
If \((r, \theta) \in \overline{D}_1^{k_0}\), then

\begin{equation}
2k_0\theta_0 - 2(k_0 - 1)\pi \leq \theta \leq (2k_0 + 1)\theta_0 - 2(k_0 - 1)\pi.
\end{equation}

This implies, using (9), that \(\theta \geq \theta_0\). But \((r, \theta) \in \overline{D}_1\) implies \(\theta \leq \theta_0\). Hence \(\theta = \theta_0\). But then \((r, \theta_0)\) cannot belong to \(\overline{D}_1\), a contradiction. This proves identity (8).

Hence we have extended \(u_1|_{\Omega\setminus D_1}\) harmonically into \(\Omega \setminus \{0\}\). Since we already know \(u_1|_{\Omega\setminus D_1}\) has a harmonic extension to \(B_{\epsilon}(0)\), we conclude that \(u_1|_{\Omega\setminus D_1}\) has a harmonic extension to \(\Omega\). Let \(v\) denote this harmonic extension to \(\Omega\). Because of the uniqueness in the Dirichlet problem, \(v = u_1\) in \(D_1\). Hence the transmission conditions (3) imply \(\frac{\partial u_1}{\partial n} = 0\) on \(\partial D_1\), so \(u_1\) is a constant in \(\Omega\). Hence \(g \equiv 0\), a contradiction.

Lemma 3 below and the ideas of Theorem 1 give us the following result in \(R^n\).

**Theorem 2.** Let \(n \geq 2\). Assume \(D_1\) and \(D_2\) are convex polyhedra. Under the assumption on Theorem 1, \(D_1 = D_2\).

Here we include the proof of Lemma 3, the ideas of which were already in Friedman and Isakov [1].

**Lemma 3.** Let \(n \geq 2\). Assume \(u\) is a solution to (1). Let \(\gamma\) be an edge of \(D\) with angle \(\theta_1\). Suppose that there is a \(Q \in \gamma\) with \(B_{r_0}(Q) \subset \Omega\) for some small \(r_0 > 0\) such that \(u|_{B_{r_0}(Q)\setminus D}\) has harmonic continuation to the whole \(B_{r_0}(Q)\). Then there is a rotation \(R_{2\theta_1}\) on the 2-dimensional plane \(E\) perpendicular to \(\gamma\) at \(Q\) such that \(u(x) = u(R_{2\theta_1}(x))\) for all \(x \in B_{\epsilon}\) and some \(\epsilon < r_0\).

**Proof.** We will give the proof for \(n \geq 3\), with the obvious changes for \(n = 2\). We can assume \(Q\) is the origin of \(R^n\), \(\gamma\) is the \((n - 2)\)-dimensional segment

\[\gamma = \{(x_1, x_2, \ldots, x_n) \in R^n : x_1 = x_2 = 0\} \cap B_1(0)\]
obtained as the intersection \(F_1 \cap F_2\) of two faces of the polygon, \(E\) is the two-dimensional plane \(x_3 = \cdots = x_n = 0\) in which we introduce polar coordinates \((r, \theta)\) and

\[E \cap D \cap B_{r_0}(0) = \{(r, \theta) : 0 < r < r_0, 0 < \theta < \theta_1\}\]

We will first observe that if \(u|_{B_{r_0}\setminus D}\) has a harmonic continuation to \(B_{r_0}(0)\), then there is a small \(\epsilon > 0\) with \(\epsilon < r_0\) such that \(u|_{B_{\epsilon}\cap D}\) has a harmonic continuation into the whole \(B_{\epsilon}(0)\). To see this, let \(u^{\text{ext}}\) denote the harmonic extension of \(u|_{B_{r_0}\setminus D}\) to \(B_{r_0}\), and consider the Cauchy problem

\begin{equation}
\begin{cases}
\Delta w = 0 & \text{in } B_{\epsilon}(0), \\
w = u, \quad \frac{\partial w}{\partial n} = \frac{1}{2} \frac{\partial u^{\text{ext}}}{\partial n} & \text{on } \pi_1 \cap B_{\epsilon}(0),
\end{cases}
\end{equation}

where \(\pi_1\) is the \((n - 1)\)-dimensional plane containing the face \(F_1\). Since \(u^{\text{ext}}\) is analytic on \(B_{\epsilon}(0)\), by the Cauchy-Kovalevski Theorem there exists a unique analytic solution to (12) on \(B_{\epsilon}(0)\) if \(\epsilon\) is small enough. But \(u|_{B_{\epsilon}\cap D}\) satisfies the same equation in \(D \cap B_{\epsilon}(0)\) with the same Cauchy data on \(F_1 \cap B_{\epsilon}(0)\). Therefore, \(w = u\) in \(B_{\epsilon} \cap D\) by the uniqueness of harmonic extension, and we will denote \(u^{\text{int}} = w\) in \(B_{\epsilon}\).
Consider now
\[ v_1 = \mathbf{u}^{\text{ext}} - \mathbf{u}^{\text{int}}, \quad v_2 = \mathbf{u}^{\text{ext}} - 2\mathbf{u}^{\text{int}}. \]

Because of the continuity of the solution across the interfaces and the jump relations (3) on \( \partial D \), we have
\[
(13) \quad v_1 = 0 \quad \text{and} \quad \frac{\partial v_2}{\partial \theta} = 0 \quad \text{for} \quad 0 < r < \varepsilon, \quad \theta = 0, \theta_1.
\]

Since \( v_1, v_2 \) are harmonic in \( B_\varepsilon(0) \), we can write for every \( x' = (x_3, \ldots, x_n) \in \mathbb{R}^{n-2} \)
\[
v_1(r, \theta, x') = \sum_{k=0}^{\infty} \left( a_k^1(x')\cos(k\theta) + b_k^1(x')\sin(k\theta) \right) r^k,
\]
\[
v_2(r, \theta, x') = \sum_{k=0}^{\infty} \left( a_k^2(x')\cos(k\theta) + b_k^2(x')\sin(k\theta) \right) r^k.
\]

From (13), we obtain
\[
a_k^1(x') = 0, \quad b_k^1\sin(k\theta_1) = 0 \quad \text{for} \quad k = 1, 2, \ldots, \\
a_k^2(x')\sin(k\theta_1) = 0, \quad b_k^2(x') = 0 \quad \text{for} \quad k = 1, 2, \ldots.
\]

Hence \( v_i(r, \theta + 2\theta_1, x') = v_i(r, \theta, x') \) for \( i = 1, 2 \). Therefore, the same hold for \( \mathbf{u}^{\text{ext}} \) and \( \mathbf{u}^{\text{int}} \), and the conclusion of Lemma 3 follows.

Remark. When \( D_1, D_2 \) in Theorem 1 are general polyhedra, the above proof shows convex hull\((D_1) = \text{convex hull}(D_2)\).

References


Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

G.A.R.C. Seoul National University, Seoul, Korea

Current address: Department of Mathematics, Pohang Institute of Science and Technology, Pohang, 790-600 Korea