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# THE INVERSE CONDUCTIVITY PROBLEM WITH ONE MEASUREMENT: UNIQUENESS FOR CONVEX POLYHEDRA

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ABSTRACT. Let $\Omega$ denote a smooth domain in $R^n$ containing the closure of a convex polyhedron $D$. Set $\chi_D$ equal to the characteristic function of $D$. We find a flux $g$ so that if $u$ is the nonconstant solution of $\operatorname{div}((1+\chi_D)\nabla u) = 0$ in $\Omega$ with $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$, then $D$ is uniquely determined by the Cauchy data $g$ and $f \equiv u/\partial\Omega$.

## INTRODUCTION

Let $\Omega$ be a bounded domain in $R^n$, $n \geq 2$, with a connected boundary and $D$ a subdomain in $\Omega$. Assume both $\Omega$ and $D$ are conductors of electricity. We consider the following question: Can we set up a magnetic field $\vec{E}$ surrounding $\Omega$ with a known flux $g$ across $\partial\Omega$ so that calculating the potential of the field on $\partial\Omega$ will determine $D$?

Writing $\vec{E} = \nabla u$ we have

$$\begin{aligned} L_\gamma u = {} & \operatorname{div}(\gamma(x)\nabla u) = 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} & = g \quad \text{on } \partial\Omega, \end{aligned}$$

where $\gamma(x)$ is the conductivity and $\frac{\partial u}{\partial n}$ denotes the normal derivative of $u$ on $\partial\Omega$. The question now becomes: Can we choose $g$ so that $g$ and $f = u|_{\partial\Omega}$ uniquely determine $D$?

For the sake of definiteness we take $\gamma(x) = 1 + \chi_D(x)$, where $\chi_D$ denotes the characteristic function of $D$. In this case, Friedman and Isakov [1] proved that there is a flux $g$ so that $g$ and $f$ uniquely determine $D$ if $D$ is assumed to be a convex polyhedron situated away from the boundary of $\Omega$, that is, if

$$\operatorname{diam}(D) < \operatorname{dist}(D, \partial\Omega).$$

In this paper we are able to remove this extra condition. Specifically, we prove the following uniqueness result: There exists a function $g$ defined on

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$\partial\Omega$ such that if $D_1, D_2$ are convex polyhedra compactly contained in $\Omega$ and $u_i$, $i = 1, 2$, is the solution of

$$\operatorname{div}((1+\chi_{D_i}(x))\nabla u_i(x)) = 0 \quad \text{in } \Omega,$$
$$\int_\Omega u_i\,dx = 0 \quad \text{and} \quad \frac{\partial u_i}{\partial n}|_{\partial\Omega} = g,$$

then if $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$, we conclude $D_1 = D_2$. We prove this result in §3 (Theorem 1) for the case of convex polygons. The proof for convex polyhedra in $R^n$ follows from Lemma 3 using similar arguments.

## 2. Preliminaries

We recall that by definition a convex polyhedron $\mathscr{P}$ in $R^n$ is a finite intersection of half spaces. That is, if

$$H_i = \{x \in R^n : (x - x_i)\cdot n_i > 0\}$$

for some points $x_1, x_2, \dots, x_k \in R^n$ and unit vectors $n_1, n_2, \dots, n_k \in R^n$, then

$$\mathscr{P} = H_1 \cap H_2 \cap \cdots \cap H_k.$$

An edge $\gamma$ of the polyhedron $\mathscr{P}$ is the intersection of two faces, i.e., the intersection of two hyperplanes used in the definitions of $\mathscr{P}$. A polyhedron $D$ in $R^n$ is an open connected set that is the union of convex polyhedrons.

Given a bounded $C^2$ domain $\Omega$ in $R^n$ and a polyhedron $D$ compactly supported in $\Omega$, let us consider the Neumann problem

$$(1)\qquad \begin{cases} Lu = \operatorname{div}((1+\chi_D(x))\nabla u) = 0 & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega, \end{cases}$$

where $n$ is the unit normal to the boundary and $u$ is normalized by $\int_\Omega u = 0$.

By a weak solution $u$ to $Lu = 0$ we mean a function $u \in L^2(\Omega)$ whose partial derivatives $u_{x_i}$, in the distributional sense, belong to $L^2(\Omega)$ and such that

$$(2)\qquad \sum_{i=1}^n \int_\Omega (1+\chi_D)\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i}\,dx = 0$$

for all $\phi \in C_0^\infty(\Omega)$, the class of $C^\infty$ functions with compact support in $\Omega$.

It is well known that if $g \in L^\infty(\partial\Omega)$, then there exists a unique solution $u$ to (1) such that $u \in C^\alpha(\bar\Omega)$ for some $\alpha > 0$.

If we set

$$u^e = u|_{\Omega\setminus D} \quad \text{and} \quad u^i = u|_D,$$

then $\Delta u^e = 0$ in $\Omega\setminus\bar D$, $\Delta u^i = 0$ in $D$, and $u^e = u^i$ on $\partial D$.

Also, by integrating by parts in (1) we get

$$(3)\qquad \frac{\partial u^e}{\partial n} = 2\frac{\partial u^i}{\partial n}$$

on the smooth part of $\partial D$.

## 3. The main theorem

Let $\Omega \subset R^n$ be a bounded convex $C^2$ domain in $R^n$ with a connected boundary. Let $D_1, D_2$ be two polyhedra compactly supported in $\Omega$. For

$i = 1, 2$ consider

$$\tag{4} \begin{cases} L_{\gamma_i} u_i = \operatorname{div}\,((1+\chi_{D_i}(x))\nabla u_i) = 0 & \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial n} = g \quad \text{on } \partial\Omega, \end{cases}$$

where we also take $\int_\Omega u_i\,dx = 0$.

Let us fix a $L^\infty(\partial\Omega)$ function $g$ such that there is no harmonic function $u$ on any neighborhood $\mathscr{V}$ of any point of $\partial\Omega$ with $g = \nabla u \cdot n$ on $\mathscr{V} \cap \partial\Omega$. For example, let us take a dense sequence of points $\{p_k\}$ on the surface boundary $\partial\Omega$ and consider the function

$$\tag{5} g(x) = \sum_{k=1} 2^{-k}|x - p_k| - c,$$

where $x \in R^n$ and $c$ is chosen so that $\int_{\partial D} g\,d\sigma = 0$. The function $g$ is Lipschitz but not $C^1$ on any surface ball $B(p) \cap \partial\Omega$. If $u$ is harmonic in $B(p)$, then $\nabla u \cdot n$ is $C^1(B(p)) \cap \partial\Omega$, so we cannot have $\nabla u \cdot n = g$ there.

**Theorem 1.** *Let $D_1$, $D_2$ be convex polygons in $R^2$. Assume $u_i$, $i = 1, 2$, are the solutions to the Neumann problem* (4) *with $g$ as in* (5). *If $u_1 = u_2$ on $\partial\Omega$, then $D_1 = D_2$.*

*Proof.* If $D_1$ and $D_2$ have the same corners, there is nothing to be proved. Let us assume that $P$ is a corner of $D_1$ but $P$ is not a corner of $D_2$.

We can also assume, for simplicity, that $P$ is the origin of $R^2$ and, in polar coordinates, if $B_{r_0}$ is a small ball centered at the origin,

$$B_{r_0} \cap D_1 = \{(r, \theta) : 0 < r < r_0,\ 0 < \theta < \theta_0\}$$

and

$$B_{r_0} \cap D_2 = \varnothing.$$

Note that $0 < \theta_0 < \pi$ because of the convexity of $D_1$.

Since $u_1 = u_2$ on $\partial\Omega$ and also $\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = g$ on $\partial\Omega$, by the unique continuation property for harmonic functions it follows that $u_1 = u_2$ on the connected component of $\Omega \setminus (D_1 \cup D_2)$ near the boundary $\partial\Omega$. Since $u_2$ is harmonic on $B_{r_0}$, $u_1|_{B_{r_0}\setminus D_1}$ has a harmonic extension to the whole ball $B_{r_0}$. (In fact, this harmonic extension is $u_2$ itself.) Let us call $u_1^{\text{ext}}$ this harmonic extension.

From Lemma 3 below, in polar coordinates, we have

$$\tag{6} u_1^{\text{ext}}(r, \theta) = u_1^{\text{ext}}(r, \theta \pm 2\theta_0)$$

for all $(r, \theta) \in B_\epsilon$, for some $0 < \epsilon < r_0$.

Let us represent by $\tilde{u}_1$ the rotation of $u_1$ by the angle $2\theta_0$, that is,

$$\tilde{u}_1(r, \theta) \equiv u_1(r, \theta - 2\theta_0),$$

and let us denote by $\tilde{\Omega}$, $\tilde{D}_1$ the rotated corresponding domains; that is,

$$\tilde{\Omega} = \{(r, \theta + 2\theta_0) : (r, \theta) \in \Omega\},$$
$$\tilde{D}_1 = \{(r, \theta + 2\theta_0) : (r, \theta) \in D_1\}.$$

Since

$$u_1(r, \theta) = u_1^{\text{ext}}(r, \theta) \qquad \text{for } (r, \theta) \in B_\epsilon \setminus D_1,$$
$$\tilde{u}_1(r, \theta) = u_1^{\text{ext}}(r, \theta - 2\theta_0) \quad \text{for } (r, \theta) \in B_\epsilon \setminus \tilde{D}_1,$$

it follows from (6) that

$$u_1 = \tilde{u}_1 \quad \text{in } (B_\epsilon \setminus D_1) \cap (B_\epsilon \setminus \tilde{D}_1).$$

Now, by unique continuation, we have $u_1 = \tilde{u}_1$ on all open connected components near the origin of $(\Omega \cap \tilde{\Omega}) \setminus (D_1 \cup \tilde{D}_1)$.

Since $\theta_0 < \pi$, there are $\theta_1$ and $\delta > 0$ such that if $\theta_1 < \phi < \theta_1 + \delta$, then the ray

$$T_\phi = \{(r, \theta) : r > 0, \theta = \phi\}$$

does not intersect $D_1 \cup \tilde{D}_1$. Let $Q_\phi$ and $\tilde{Q}_\phi$ be the points on $\partial\Omega$ and $\partial\tilde{\Omega}$ respectively such that

$$(0, Q_\phi) \subset \Omega \cap T_\phi \quad \text{and} \quad (0, \tilde{Q}_\phi) \subset \tilde{\Omega} \cap T_\phi,$$

where $(0, Q_\phi)$ denotes the open line segment joining the origin $0$ with $Q_\phi$ and $(0, \tilde{Q}_\phi)$ the open line segment joining $0$ with $\tilde{Q}_\phi$. We will prove

$$Q_\phi = \tilde{Q}_\phi \quad \text{for all } \theta_1 < \phi < \theta_1 + \delta.$$

If not, we may assume $(0, Q_\phi) \subset (0, \tilde{Q}_\phi)$. Since $u_1$ and $\tilde{u}_1$ coincide on $(0, Q_\phi)$ and $\tilde{u}_1$ is harmonic in a neighborhood of $Q_\phi$, a contradiction follows because $u_1$ cannot be harmonically extended across any small neighborhood of the point $Q_\phi$ because of the bad boundary data $g$. This proves that if

$$\mathscr{S} = \bigcup_{\theta_1 < \phi < \theta_1 + \delta} (0, Q_\phi),$$

then

$$\Omega \cap \mathscr{S} = \tilde{\Omega} \cap \mathscr{S} = \mathscr{S}, \qquad \bigcup_{\theta_1 < \phi < \theta_1 + \delta} \{Q_\phi\} \subset \partial\Omega \cap \partial\tilde{\Omega}.$$

Now we will prove $\partial\Omega = \partial\tilde{\Omega}$. Let $P_0 \in \bigcup_{\theta_1 < \phi < \theta_1 + \delta} \{Q_\phi\}$, and let $d = \frac{1}{2}\operatorname{dist}(D_1, \partial\Omega)$. Now we define

$$\begin{aligned} s(P_0) = \sup_{0 \le r \le d} \{r : B_r(P_0) & \text{ has the property ; for every } P \in B_r(P_0) \cap \partial\Omega \\ & \text{there exists a ball } B_\epsilon(P) \text{ such that } B_\epsilon(P) \cap \Omega = B_\epsilon(P) \cap \tilde{\Omega} \\ & \text{and } B_\epsilon(P) \cap \Omega^c = B_\epsilon(P) \cap \tilde{\Omega}^c\}. \end{aligned}$$

We claim $s(P_0) = d$. Suppose $s < d$. We can pick $\bar{P} \in \partial B_s(P_0) \cap \partial\Omega$. Clearly

$$\bar{P} \in \partial\Omega \cap \partial\tilde{\Omega}, \qquad \bar{P} \in \partial(\Omega \cap \tilde{\Omega}).$$

Recall $\Omega$ is convex. We can choose a small ball $B(\bar{P})$ such that in local coordinates there exist functions $\psi$ and $\tilde{\psi}$ such that

$$B(\bar{P}) \cap \Omega = \{(x, y) : y < \psi(x)\} \cap B(\bar{P})$$

and

$$B(\bar{P}) \cap \tilde{\Omega} = \{(x, y) : y < \tilde{\psi}(x)\} \cap B(\bar{P}).$$

From the uniqueness of harmonic continuation, it is easy to see that

$$u_1 = \tilde{u}_1 \quad \text{in } B_s(P_0) \cap \Omega \cap \tilde{\Omega},$$

so

$$u_1 = \tilde{u}_1 \quad \text{in } B(\bar{P}) \cap \Omega \cap \tilde{\Omega}.$$

Let $\bar{P} = (x_0, \psi(x_0))$. Then there is a small $\epsilon_1 > 0$ such that

$$(x, \psi(x)), (x, \tilde{\psi}(x)) \in B(\bar{P}) \quad \text{whenever } |x - x_0| < \epsilon_1.$$

We will show $B_{\epsilon_1}(\bar{P}) \cap \Omega = B_{\epsilon_1}(\bar{P}) \cap \tilde{\Omega}$. It suffices to show

$$\psi(x) = \tilde{\psi}(x) \quad \text{for } |x - x_0| < \epsilon_1.$$

If not, there is an $x$ with $|x-x_0| < \epsilon_1$ such that $\psi(x) < \tilde{\psi}(x)$ or $\tilde{\psi}(x) < \psi(x)$. Suppose $\psi(x) < \tilde{\psi}(x)$. Then $(x, \psi(x)) \in \tilde{\Omega} \cap B(\bar{P})$, and therefore there is a small ball $B_{\epsilon_2}(Q)$ where $Q = (x, \psi(x))$ such that $B_{\epsilon_2}(Q) \subset B(\bar{P}) \cap \tilde{\Omega}$. Since

$$u_1 = \tilde{u}_1 \quad \text{in } B_{\epsilon_2}(Q) \cap \Omega \subset B(\bar{P}) \cap \Omega \cap \tilde{\Omega}$$

and $\tilde{u}_1$ is harmonic in $B_{\epsilon_2}(Q)$, $u_1$ has a harmonic extension to $B_{\epsilon_2}(Q)$, and this is not possible because of the bad boundary data $g$. This proves $s(P_0) = d$.

Next we pick $P_1 \in \partial\Omega \cap \partial B_{d/2}(P_0)$. Then the same arguments as above show that $s(P_1) = d$, and by repeating this process we conclude $\partial\Omega = \partial\tilde{\Omega}$ or $\Omega = \tilde{\Omega}$. Hence $u_1$ has a harmonic extension to

$$(\Omega \setminus \overline{D_1}) \cup (\Omega \setminus \overline{\tilde{D}_1}).$$

We can repeat the same arguments as before for the $k$-rotation of $u_1$

$$\tilde{u}_1^k(r, \theta) = u_1(r, \theta - 2k\theta_0)$$

for $k = 0, 1, 2, \ldots$ in such a way that any consecutive two rotations $\tilde{u}_1^k$ and $\tilde{u}_1^{k+1}$ play the role of $u_1$ and $\tilde{u}_1$ respectively, concluding finally that $u_1$ has a harmonic extension to

$$(\Omega \setminus \overline{D_1}) \cup (\Omega \setminus \overline{\tilde{D}_1}) \cup \cdots \cup (\Omega \setminus \overline{\tilde{D}_1^k}).$$

Here $\tilde{D}_1^k = \{(r, \theta + 2k\theta_0) : (r, \theta) \in D_1\}$.

We claim that this last set is $\Omega \setminus \{0\}$ for some finite $k$. Since $2\theta_0 < 2\pi$, there is the positive integer $k_0$ such that

(7) $\quad (2k_0 + 1)\theta_0 < 2k_0\pi$ and $(2k + 1)\theta_0 \geq 2k\pi \quad$ for $k = 0, 1, \ldots, k_0 - 1$.

We will show

$$\Omega = (\Omega \setminus \overline{D_1}) \cup \left(\Omega \setminus \overline{\tilde{D}_1}\right) \cup \left(\Omega \setminus \overline{\tilde{D}_1^{k_0}}\right) \cup \{0\}. \tag{8}$$

In fact, if that is not true, there is an $(r, \theta) \in \Omega$ such that

$$(r, \theta) \in \bar{D}_1 \cap \overline{\tilde{D}_1} \cap \overline{\tilde{D}_1^{k_0}}.$$

From (7) we can get

$$0 \leq (2k_0 - 1)\theta_0 - 2(k_0 - 1)\pi < (2k_0 + 1)\theta_0 - 2(k_0 - 1)\pi < 2\pi, \tag{9}$$

where the first inequality comes from the second statement in (7) with $k = k_0 - 1$ and the last inequality comes from the first statement in (7).

Also, we can rewrite the first inequality in (9) as

$$\theta_0 \leq 2k_0\theta_0 - 2(k_0 - 1)\pi. \tag{10}$$

If $(r, \theta) \in \overline{\tilde{D}_1^{k_0}}$, then

$$2k_0\theta_0 - 2(k_0 - 1)\pi \le \theta \le (2k_0 + 1)\theta_0 - 2(k_0 - 1)\pi. \tag{11}$$

This implies, using (9), that $\theta \ge \theta_0$. But $(r, \theta) \in \overline{D_1}$ implies $\theta \le \theta_0$. Hence $\theta = \theta_0$. But then $(r, \theta_0)$ cannot belong to $\overline{\tilde{D}_1}$, a contradiction. This proves identity (8).

Hence we have extended $u_1|_{\Omega\backslash D_1}$ harmonically into $\Omega\backslash\{0\}$. Since we already know $u_1|_{\Omega\backslash D_1}$ has a harmonic extension to $B_\epsilon(0)$, we conclude that $u_1|_{\Omega\backslash D_1}$ has a harmonic extension to $\Omega$. Let $v$ denote this harmonic extension to $\Omega$. Because of the uniqueness in the Dirichlet problem, $v = u_1$ in $D_1$. Hence the transmission conditions (3) imply $\frac{\partial u_1}{\partial n} = 0$ on $\partial D_1$, so $u_1$ is a constant in $\Omega$. Hence $g \equiv 0$, a contradiction.

Lemma 3 below and the ideas of Theorem 1 give us the following result in $R^n$.

**Theorem 2.** *Let $n \ge 2$. Assume $D_1$ and $D_2$ are convex polyhedra. Under the assumption on Theorem* 1, *$D_1 = D_2$.*

Here we include the proof of Lemma 3, the ideas of which were already in Friedman and Isakov [1].

**Lemma 3.** *Let $n \ge 2$. Assume $u$ is a solution to* (1). *Let $\gamma$ be an edge of $D$ with angle $\theta_1$. Suppose that there is a $Q \in \gamma$ with $B_{r_0}(Q) \subset \Omega$ for some small $r_0 > 0$ such that $u|_{B_{r_0}(Q)\backslash D}$ has harmonic continuation to the whole $B_{r_0}(Q)$. Then there is a rotation $R_{2\theta_1}$ on the 2-dimensional plane $E$ perpendicular to $\gamma$ at $Q$ such that $u(x) = u(R_{2\theta_1}(x))$ for all $x \in B_\epsilon$ and some $\epsilon < r_0$.*

*Proof.* We will give the proof for $n \ge 3$, with the obvious changes for $n = 2$. We can assume $Q$ is the origin of $R^n$, $\gamma$ is the $(n-2)$-dimensional segment

$$\gamma = \{(x_1, x_2, \dots, x_n) \in R^n : x_1 = x_2 = 0\} \cap B_1(0)$$

obtained as the intersection $F_1 \cap F_2$ of two faces of the polygon, $E$ is the two-dimensional plane $x_3 = \cdots = x_n = 0$ in which we introduce polar coordinates $(r, \theta)$ and

$$E \cap D \cap B_{r_0}(0) = \{(r, \theta) : 0 < r < r_0, 0 < \theta < \theta_1\}.$$

We will first observe that if $u|_{B_{r_0}\backslash D}$ has a harmonic continuation to $B_{r_0}(0)$, then there is a small $\epsilon > 0$ with $\epsilon < r_0$ such that $u|_{B_\epsilon \cap D}$ has a harmonic continuation into the whole $B_\epsilon(0)$. To see this, let $u^{\text{ext}}$ denote the harmonic extension of $u|_{B_{r_0}\backslash D}$ to $B_{r_0}$, and consider the Cauchy problem

$$\begin{cases} \Delta w = 0 \quad \text{in } B_\epsilon(0), \\ w = u, \ \dfrac{\partial w}{\partial n} = \dfrac{1}{2}\dfrac{\partial u^{\text{ext}}}{\partial n} \quad \text{on } \pi_1 \cap B_\epsilon(0), \end{cases} \tag{12}$$

where $\pi_1$ is the $(n-1)$-dimensional plane containing the face $F_1$. Since $u^{\text{ext}}$ is analytic on $B_\epsilon(0)$, by the Cauchy-Kovalevski Theorem there exists a unique analytic solution to (12) on $B_\epsilon(0)$ if $\epsilon$ is small enough. But $u|_{B_\epsilon \cap D}$ satisfies the same equation in $D \cap B_\epsilon(0)$ with the same Cauchy data on $F_1 \cap B_\epsilon(0)$. Therefore, $w = u$ in $B_\epsilon \cap D$ by the uniqueness of harmonic extension, and we will denote $u^{\text{int}} = w$ in $B_\epsilon$.

Consider now

$$v_1 = u^{\text{ext}} - u^{\text{int}}, \qquad v_2 = u^{\text{ext}} - 2u^{\text{int}}.$$

Because of the continuity of the solution across the interfaces and the jump relations (3) on $\partial D$, we have

$$v_1 = 0 \quad \text{and} \quad \frac{\partial v_2}{\partial \theta} = 0 \qquad \text{for } 0 < r < \tilde{\epsilon},\ \theta = 0, \theta_1. \tag{13}$$

Since $v_1$, $v_2$ are harmonic in $B_{\tilde{\epsilon}}(0)$, we can write for every $x' = (x_3, \dots, x_n) \in R^{n-2}$

$$v_1(r, \theta, x') = \sum_{k=0}^{\infty} \left(a_k^1(x')\cos(k\theta) + b_k^1(x')\sin(k\theta)\right) r^k,$$

$$v_2(r, \theta, x') = \sum_{k=0}^{\infty} \left(a_k^2(x')\cos(k\theta) + b_k^2(x')\sin(k\theta)\right) r^k.$$

From (13), we obtain

$$a_k^1(x') = 0, \quad b_k^1 \sin(k\theta_1) = 0 \qquad \text{for } k = 1, 2, \dots,$$
$$a_k^2(x')\sin(k\theta_1) = 0, \quad b_k^2(x') = 0 \qquad \text{for } k = 1, 2, \dots.$$

Hence $v_i(r, \theta + 2\theta_1, x') = v_i(r, \theta, x')$ for $i = 1, 2$. Therefore, the same hold for $u^{\text{ext}}$ and $u^{\text{int}}$, and the conclusion of Lemma 3 follows.

*Remark.* When $D_1$, $D_2$ in Theorem 1 are general polyhedra, the above proof shows convex hull$(D_1)$ = convex hull$(D_2)$.

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