EXPONENETION IS HARD TO AVOID

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Abstract. Let \( \mathcal{R} \) be an O-minimal expansion of the field of real numbers. If \( \mathcal{R} \) is not polynomially bounded, then the exponential function is definable (without parameters) in \( \mathcal{R} \). If \( \mathcal{R} \) is polynomially bounded, then for every definable function \( f : \mathbb{R} \rightarrow \mathbb{R}, f \) not ultimately identically 0, there exist \( c, r \in \mathbb{R}, c \neq 0 \), such that \( x \mapsto x^r : (0, +\infty) \rightarrow \mathbb{R} \) is definable in \( \mathcal{R} \) and \( \lim_{x \rightarrow +\infty} f(x)/x^r = c \).

In the following, let \( \mathcal{R} := (\mathbb{R}, <, 0, 1, +, \cdot, \ldots) \) be an expansion of the ordered field of real numbers. "Definable" means first-order definable in \( \mathcal{R} \) with parameters from \( \mathbb{R} \). A function \( f : X \rightarrow \mathbb{R}, X \subseteq \mathbb{R} \), is said to be definable if its graph is definable. We may, whenever convenient, assume that we deal with totally defined functions by setting partial functions equal to 0 off their domain of definition. For the rest of this note, all functions mentioned are of one variable.

We say that \( \mathcal{R} \) is polynomially bounded if, for every definable function \( f \), there exists \( N \in \mathbb{N} \) such that ultimately \( |f(x)| \leq x^N \). ("Ultimately" abbreviates "for all sufficiently large positive \( x \).") We say that \( \mathcal{R} \) is O-minimal if the definable subsets of \( \mathbb{R} \) are just the finite unions of intervals of all kinds, including singletons. (For general facts about O-minimal structures, see [PS] and [KPS].)

Given definable functions \( f \) and \( g \) such that ultimately \( g(x) \neq 0 \), we write \( f \sim g \) if \( \lim_{x \rightarrow +\infty} f(x)/g(x) = 1 \).

Theorem. Let \( \mathcal{R} \) be O-minimal and not polynomially bounded. Then the exponential function is definable.

Proof. By [vdD], every definable function is ultimately differentiable and either constant or strictly monotone. Thus, the germs at \( +\infty \) of definable functions form a Hardy field. In particular, every definable function has a limit in \( \mathbb{R} \cup \{-\infty, +\infty\} \) at \( +\infty \). Since \( \mathcal{R} \) is not polynomially bounded, there exists a definable function \( f \) such that, for all \( n \in \mathbb{N}, \lim_{x \rightarrow +\infty} f(x)/x^n = +\infty \). By Proposition 6 of [R], there exists a definable function \( h \) with \( h' \sim f'/f \). Since \( f \) is ultimately strictly increasing, \( f \) has a definable compositional inverse \( f^{-1} \).

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for large positive $x$. Then $(h \circ f^{-1})' \sim 1/x$ (where $x$ denotes the identity function). Let $g = h \circ f^{-1}$, so $g$ is definable and $g' \sim 1/x$.

We now show that the existence of a definable function $g$ with $g' \sim 1/x$ implies that log is definable.

First, we claim that, for all $t > 0$, $t \neq 1$, $\lim_{x \to +\infty} (g(tx) - g(x))$ is finite and nonzero.

Suppose $t > 1$. By the Mean Value Theorem, for large $x$ we have

$$(g(xt) - g(x))/(t - 1) = xg'(\xi(x)), \quad \text{with } x < \xi(x) < xt.$$ 

Note then that $1/t < x/\xi(x) < 1$. Since $\lim_{x \to +\infty} xg'(x) = 1$, ultimately we have $|xg'(\xi(x)) - x/\xi(x)| < x/(2t\xi(x))$, and thus

$$|(g(xt) - g(x))/(t - 1) - x/\xi(x)| < 1/(2t).$$

So ultimately $(t - 1)/(2t) < g(xt) - g(x) < (t - 1)(1 + 1/(2t))$. Now $x \mapsto g(xt) - g(x)$ is definable, so $\lim_{x \to +\infty} (g(xt) - g(x))$ exists, and the claim holds.

For $t \in (0, 1)$, note that

$$\lim_{x \to +\infty} (g(xt) - g(x)) = - \lim_{x \to +\infty} (g(x/t) - g(x)).$$

We have now shown that there is a definable function $G : (0, +\infty) \to \mathbb{R}$ with $G(t) = \lim_{x \to +\infty} (g(xt) - g(x))$. Note that $G(1) = 0$, so $G$ is nonconstant. It is easy to check that $G(st) = G(s) + G(t)$ for all $s, t > 0$. By O-minimality, $G$ is ultimately differentiable. By elementary real analysis, we have $G(t) = G'(1) \log t$ for all $t > 0$. Since $G'(1) \neq 0$, log is definable, and thus exp is as well. □

Remark. The exponential function is thus even definable without parameters from $\mathbb{R}$, using the fact that the exponential function is the unique differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 1$ and $f' = f$; see [vdD].

The first known example of an O-minimal expansion of the ordered field of real numbers that is not polynomially bounded is due to Wilkie, who has recently shown (in [W1] and [W2]) that $(\mathbb{R}, <, 0, 1, +, \cdot, \exp)$ is O-minimal.

**Proposition.** Let $\mathcal{R}$ be O-minimal and polynomially bounded. Let $f$ be definable and not ultimately identically 0. Then there exist $c, r \in \mathbb{R}$, $c \neq 0$, such that $f \sim cx^r$ and $x \mapsto x^r : (0, +\infty) \to \mathbb{R}$ is definable.

**Proof.** Since $\mathcal{R}$ is polynomially bounded, by the remarks immediately preceding Proposition 4 of [R], there exist $c, r$ such that the desired asymptotic condition holds. Then $x \mapsto x^r : (0, +\infty) \to \mathbb{R}$ is definable (as noted in [P]), since, for all $x > 0$, $\lim_{y \to +\infty} f(xy)/f(y) = x^r$. □

**References**


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