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ELEMENTARY PROOF OF FURSTENBERG'S DIOPHANTINE RESULT

MICHAEL D. BOSHERNITZAN

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ABSTRACT. We present an elementary proof of a diophantine result (due to H. Furstenberg) which asserts (in a special case) that for every irrational α the set $\{2^m 3^n \alpha | m, n \geq 0\}$ is dense modulo 1. Furstenberg's original proof employs the theory of disjointness of topological dynamical systems.

1. INTRODUCTION

Throughout the paper by a *semigroup* we mean an infinite subset of positive integers which is closed under multiplication. Two integers p, q are called multiplicitively independent if both are ≥ 2 and the ratio of their logarithms $(\log p)/(\log q)$ is irrational. The equivalent requirement is that p and q should not be integral powers of a single integer.

Given a semigroup $S = \{s_1, s_2, \dots\}$, $s_1 < s_2 < s_3 < \cdots$, one easily verifies that the following three conditions are equivalent.

(1a) There is a pair of multiplicatively independent integers in S.
(1b) One cannot represent all s_i as integral powers of a single integer.
(1c) $\lim_{i\to\infty}(s_{i+1}/s_i) = 1$.

A semigroup is said to be *nonlacunary* if the above three (equivalent) conditions are satisfied.

Denote by K the circle group $K = \mathbb{R}/\mathbb{Z} = [0, 1)$. For an integer $n \geq 1$ a set $X \subseteq K$ is said to be n-invariant if $nx \in X \pmod 1$ whenever $x \in X$. A set $X \subseteq K$ is said to be invariant under the action of a semigroup S if X is n-invariant for every $n \in S$.

In this note we sketch an elementary proof of the following diophantine result.

Theorem 1.1 [F, Theorem 4.2]. *If S is nonlacunary and α is an irrational, then $S\alpha$ is dense modulo* 1.

The above theorem is an immediate consequence of the following. (Conversely, Theorem 1.2 easily follows from Theorem 1.1 if one takes into account Lemma 2.1 in §2; we do not use this implication.)

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Theorem 1.2 [F, §4]. *Let X is a closed infinite subset of $K = \mathbb{R}/\mathbb{Z} = [0, 1)$ which is invariant under the action of a nonlacunary semigroup S. Then $X = K$.*

2. Proof of Theorem 1.2

In what follows we denote by X' the set of the limit points of X. We need the following two lemmas.

Lemma 2.1. *Under the conditions of Theorem 1.2, assume that X' contains a rational limit point. Then $X = K$.*

Lemma 2.2. *Let X be a closed nonempty subset of K which is invariant under the action of a nonlacunary semigroup S. Then X contains a rational point.*

Theorem 1.2 follows easily from the above two lemmas. Indeed, $X' \neq \varnothing$ (since X is infinite) and satisfies all the conditions of Lemma 2.2. Therefore, X' contains a rational point, and the application of Lemma 2.1 completes the proof of Theorem 1.2. Note that Lemma 2.2 is equivalent to [F, Lemma 4.2], the lemma whose proof used the disjointness argument.

3. Proof of Lemma 2.1

If $0 \in X'$, the lemma follows easily in view of (1c). For completeness, we provide the argument [F, Lemma 4.2]. Given a small $\varepsilon > 0$, take n such that for all $i \geq n$ the inequality $s_{i+1}/s_i < 1 + \varepsilon$ holds. Then take any $x \in X$ such that $0 \neq |x| < \varepsilon/s_n$, and observe that the finite set

$$\{sx | s \in S, \ s_n \leq s \leq 1/|x|\} \subset X$$

is ε-dense in K. Since $\varepsilon > 0$ is arbitrary and X is closed, $X = K$.

Now assume that X' contains a rational $r = n/t$. Take a pair p, q of multiplicatively independent integers of S (which exists since S is nonlacunary). Without loss of generality, we assume that $(n, t) = (t, p) = (t, q) = 1$ (replacing if needed r by the product of r and suitable powers of p and q; (x, y) denotes the greatest common divisor of x and y). Choose a positive integer u such that $p^u \equiv q^u \equiv 1 \pmod{t}$ (e.g., $u = \phi(t)$ where ϕ is the Euler function). The sets X and X' are clearly both p^u- and q^u-invariant, and so are their shifts $Y' = X' - r$ and $Y = X - r$, in view of the choice of u. Moreover, $0 \in Y'$. Applying the conclusion of the preceding paragraph to Y and Y', we conclude that $Y = K$. Therefore, $X = K$.

4. Proof of Lemma 2.2

Assume that X does not contain rationals. Let $\varepsilon > 0$ be given. First choose a multiplicatively independent pair p, $q \in S$. Next choose an integer $t \geq 3$ such that $\varepsilon t > 1$, $(t, p) = 1$, and $(t, q) = 1$. Finally, choose a positive integer u such that $p^u \equiv q^u \equiv 1 \pmod{t}$.

Define (inductively) a sequence of sets

$$X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_{t-1}$$

by assuming

$$X_{i+1} = \{x \in X_i | x + 1/t \in X_i \ (\mathrm{mod}\, 1)\} \quad \text{for } 0 \leq i \leq t-2. \tag{4.1}$$

Observe that, for every X_i, $0 \le i \le t-1$, the following properties hold:

(a) X_i is both p^u- and q^u-invariant.
(b) X_i is closed in K.
(c) X_i is an infinite set of irrational numbers.

Indeed, (a), (b), and (c) clearly hold for $i = 0$. We proceed by induction.

Assume that these properties take place for some i, $0 \le i \le t-2$; we have to validate the properties for $i+1$. Consider the set D_i of the differences in X_i:

$$D_i = X_i - X_i \subseteq K.$$

D_i is a closed subset of K (because X_i is compact) which is both p^u- and q^u-invariant, and 0 must be its limit point (since X_i is infinite). The semigroup generated by p^u and q^u is nonlacunary, and therefore $D_i = K$ (Lemma 2.1). It follows that the set X_{i+1} (defined by (4.1)) is nonempty.

Moreover, X_{i+1} is both p^u- and q^u-invariant (in view of the choice of u) and consequently is infinite. Finally, X_{i+1} is closed in K because X_i is.

It follows that $X_{t-1} \neq \varnothing$. Take $x_0 \in X_{t-1}$, and let $x_i = x_0 + i/t$, $0 \le i \le t-1$. The set $A = \{x_i | 0 \le i \le t-1\}$ is ε-dense in K (since $1/t < \varepsilon$), and $A \subseteq X = X_0$ (in fact, $x_i \in X_{t-i-1}$). Since $\varepsilon > 0$ is arbitrary, X is dense in K, and hence $X = K$, a contradiction with the assumption that X does not contain rationals.

5. Some related results

For a sequence $\bar{s} = \{s_i\}_{i \ge 1}$ of reals denote

$$\mathrm{NUD}(\bar{s}) = \{\alpha \in \mathbb{R} | \{\alpha s_i\} \text{ is } \textit{n}\text{ot } \textit{u}\text{niformly } \textit{d}\text{istributed in } K\}$$

and

$$\mathrm{ND}(\bar{s}) = \{\alpha \in \mathbb{R} | \{\alpha s_i\} \text{ is } \textit{n}\text{ot } \textit{d}\text{ense in } K\}$$

(with the numbers αs_i considered mod 1, i.e., as the elements of K). Clearly, for every $\bar{s}$ the inclusion $\mathrm{ND}(\bar{s}) \subseteq \mathrm{NUD}(\bar{s})$ takes place.

A classical theorem by Weyl asserts that the Lebesgue measure of $\mathrm{NUD}(\bar{s})$ is 0 for any sequence $\bar{s}$ of distinct integers. If $s_{i+1}/s_i > 1 + \varepsilon$ for some $\varepsilon > 0$ and all large i, the set $\mathrm{ND}(\bar{s})$ has Hausdorff dimension 1 and, in particular, is uncountable [P, M1, M2]. On the other hand, we have proved recently [Bos2] that if $\bar{s}$ is an unbounded sequence of positive reals such that $\lim_{i\to\infty}(s_{i+1}/s_i) = 1$ (such sequences are called sublacunary), then the Hausdorff dimension of $\mathrm{ND}(\bar{s})$ is 0. (A sharp upper bound on the Hausdorff dimension of the set $\mathrm{NUD}(\bar{s})$ in terms of the *polynomial* growth of the sequence $\bar{s}$ is given in [ET, Theorem 13].)

Note that for most (in a certain probability sense) of the sequences $\bar{s}$ of integers having a prescribed subexponential asymptotic growth we have $\mathrm{NUD}(\bar{s}) = \mathbb{Q}$, the field of rational numbers (see [Bos1, AHK, Bou]).

On the other hand, if $\bar{s}$ is the sequence of all elements of a semigroup (of integers) taken in the increasing order which grows faster than polynomials

$$\limsup_{i\to\infty} \frac{\log s_i}{\log i} = \infty,$$

then $\mathrm{NUD}(\bar{s})$ is uncountable. The requirement of transpolynomial growth cannot be weakened; for any fixed integer $n \ge 1$ we have $\mathrm{NUD}(\bar{s}) = \mathbb{Q}$ for

$\bar{s} = \{i^n\}$. Under the stronger condition that the semigroup is finitely generated, the set $\mathrm{NUD}(\bar{s})$ has Hausdorff dimension 1 [Be5].

Note that our proof of Furstenberg's theorem is close in spirit to some of the constructions of Berend [Be3] where analogous results were obtained for so-called multiplicative IP-sets of integers. Berend also dealt with the multidimensional case [Be1, Be2] and with multiplicative semigroups of algebraic numbers [Be4].

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Department of Mathematics, Rice University, Houston, Texas 77251
E-mail address: michael@math.rice.edu