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LIFTING VECTOR-VALUED MEROMORPHIC FUNCTIONS IN INFINITE DIMENSIONS

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ABSTRACT. It is shown that the lifting problem for Fréchet-valued meromorphic functions on open subsets of a (DFN)-space has a solution.

Lifting holomorphic functions in infinite dimensions has been investigated by some authors. The problem for vector-valued meromorphic functions on complex manifolds was studied in [6]. The aim of this paper is to prove that the lifting problem has a solution for Fréchet-valued meromorphic functions on open subsets of a (DFN)-space.

1. PRELIMINARIES

We shall use the standard notation from the theory of locally convex spaces as presented in the books of Pietsch [7] and Schaefer [8]. All locally convex spaces are assumed to be complex vector spaces and Hausdorff.

For a locally convex space E, we denote by $\mathscr{U}(E)$ the set consisting of all balanced convex neighbourhoods of zero in E. Let $U \in \mathscr{U}(E)$ and p_U denote the Minkowski functional on E associated to U. Then E_U denotes the completion of the canonical normed space $E/\operatorname{Ker} p_U$. The canonical map from E into E_U is written by π_U.

2. HOLOMORPHIC AND MEROMORPHIC FUNCTIONS

Let E and F be locally convex spaces and $D \subseteq E$ be open. A map $f: D \to F$ is called holomorphic if f is continuous and $f|D \cap V$ is holomorphic for every finite dimensional subspace V of E.

Now a holomorphic function $f: D_0 \to F$, where D_0 is a dense open subset of D, is said to be meromorphic on D if for every $z \in D$ there exists a neighbourhood U of z in D and holomorphic functions

$$g: U \to F, \qquad \sigma: U \to \mathbb{C}$$

such that

$$f|U \cap D_0 = g/\sigma|U \cap D_0 \quad \text{with } \sigma \not\equiv 0.$$

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Theorem 1. *Let S be a continuous linear map of a Fréchet space E onto a Fréchet space F and let D be an open subset of a (DFN)-space P. Then for every F-valued meromorphic f on D there exists an E-valued meromorphic function g on D such that $Sg = f$.*

To prove the theorem we need the following.

Lemma 2. *Every holomorphic function on an open subset of a (DFN)-space with values in a Fréchet space F can be factorized through a compact map from a Banach space into F.*

Proof. Consider an open set D in a (DFN)-space P and a holomorphic function f on D with values in F.

(i) Let $z \in D$. Since D is σ-compact [5], it can be exhausted by an increasing sequence of compact sets $\{K_n\}$, with $z \in K_1$. Let $\{V_n\}$ be a decreasing neighbourhood basis of zero in F. For each $n \geq 1$ there exists $U_n \in \mathscr{U}(P)$ and $d_n > 0$ such that

$$K_n + U_n \subseteq D \quad \text{and} \quad f(K_n + U_n) \subset d_n V_n.$$

Set

$$U = \bigcap_{n\geq 1} (K_n + U_n).$$

Since

$$K_n \cap U = \bigcap_{1\leq k\leq n} (K_k + U_k) \cap K_n$$

and since $\bigcap_{1\leq k\leq n}(K_k + U_k)$ is a neighbourhood of z in P, it follows that $K_n \cap U$ is a neighbourhood of z in K_n for every $n \geq 1$. On the other hand, since D is a k-space [5], U is a neighbourhood of z in D. From the inclusion

$$f(U) \subseteq f(K_n + U_n) \subseteq d_n V_n \quad \text{for every } n \geq 1$$

we obtain the boundedness of $f(U)$.

Consider the Taylor expansion of f at z:

$$f(z+h) = \sum_{n\geq 0} P_n f(z)(h),$$

where

$$P_n f(z)(h) = \frac{1}{2\pi i} \int_{|\lambda|=2} f(z+\lambda h)/\lambda^{n+1}\, d\lambda$$

for $h \in V$, $V \in \mathscr{U}(P)$, $z + 2V \subseteq U$. Set

$$B = \overline{\operatorname{conv}} \bigcup_{n\geq 0} P_n f(z)(V).$$

Then B is a balanced convex closed set in F and f induces a holomorphic function on $\pi_V(z+V)$ with values in the canonical Banach space $F(B)$ spanned by B.

(ii) By (i) we can find a countable open cover of D, $\{\widetilde{U}_i = z_i + U_i\}$, $U_i \in \mathscr{U}(P)$, a sequence of balanced convex bounded closed sets in F, $\{B_i\}$, and a sequence of holomorphic functions $f_i \colon \pi_{U_i}(\widetilde{U}_i) \to F(B_i)$ such that

$$f_i \pi_{U_i} = f|\widetilde{U}_i \quad \text{for every } i \geq 1.$$

Take two sequences $\{\lambda_i\} \downarrow 0$ and $\{\mu_i\} \uparrow \infty$ such that

$$B = \overline{\operatorname{conv}} \bigcup_{i \geq 1} \lambda_i B_i \quad \text{is bounded in } F$$

and

$$U = \bigcap_{i \geq 1} \mu_i U_i \in \mathscr{U}(P).$$

Such sequences exist by [4] for $\{\lambda_i\}$ and [5] for $\{\mu_i\}$. Since the canonical map from $F(\lambda_i B_i)$ into $F(B)$ is continuous for every $i \geq 1$, and $\{\pi_U(\widetilde{U}_i)\}$ is an open cover of $\pi_U(D)$ in $P/\operatorname{Ker} p_U$, it follows that the sequence $\{f_i\}$ defines an $F(B)$-valued holomorphic function g on $\pi_U(D)$ with $g\pi_U = f$.

(iii) Let $\hat{g}$ be a holomorphic extension of g to a neighbourhood $\widehat{D}_U$ of $\pi_U(D)$ in P_U. Take $V \in \mathscr{U}(P)$, $V \subseteq U$ such that the canonical map $\pi_{V,U}: P_V \to P_U$ is compact. Cover $\widehat{D}_V = \pi_{V,U}^{-1}(\widehat{D}_U)$ by a sequence of bounded open sets in P_V, $\{W_i\}$, such that $\pi_{V,U}(W_i)$ is relatively compact in $\widehat{D}_U$ for every $i \geq 1$. Then $A_i = g\pi_{V,U}(W_i)$ for every $i \geq 1$ is relatively compact in $F(B)$. Take again a sequence $\{\alpha_i\} \downarrow 0$ such that

$$A = \overline{\operatorname{conv}} \bigcup_{i \geq 1} \alpha_i A_i$$

is compact in $F(B)$. It is easy to see that $\hat{g}\pi_{V,U}: \widehat{D}_V \to F(A)$ is holomorphic. Hence f can be factorized through the compact map $F(A) \to F$. The lemma is thus proved.

3. Proof of Theorem 1

Cover D by a sequence of open subsets $\{\widetilde{U}_i\}$ of D such that $f|\widetilde{U}_i$ can be written in the form h_i/σ_i, where h_i and σ_i are holomorphic functions on $\widetilde{U}_i$ with values in F and $\mathbb{C}$, respectively. By Lemma 2, for each $i \geq 1$ we can find $U_i \in \mathscr{U}(P)$ and B_i, a balanced convex compact set in F such that h_i and σ_i are factorized through $\pi_{U_i}: \widetilde{U}_i \to \pi_{U_i}(\widetilde{U}_i)$ and $F(B_i) \to F$. Take two sequences $\{\lambda_i\} \downarrow 0$ and $\{\mu_i\} \uparrow \infty$ such that

$$B = \overline{\operatorname{conv}} \bigcup_{i \geq 1} \lambda_i B_i \quad \text{is compact in } F$$

and

$$U = \bigcap_{i \geq 1} \mu_i U_i \in \mathscr{U}(P).$$

This implies that the two sequences $\{h_i\}$ and $\{\sigma_i\}$ define a meromorphic function g on a neighbourhood $\widehat{D}_U$ of $\pi_U(D)$ in P_U with $f = g\pi_U$. By [1] there exists a balanced convex compact set A in E such that $S(A) = B$.

Cover $\widehat{D}_U$ by a sequence of open sets $\{W_j\}$ in $\widehat{D}_U$ such that for each $j \geq 1$ there exist bounded holomorphic functions g_j and σ_j on W_j with values in $F(B)$ and $\mathbb{C}$, respectively, with $g|W_j = g_j/\sigma_j$, $\sigma_j \neq 0$. Let $V \in \mathscr{U}(P)$, $V \subseteq U$ such that $T = \pi_{V,U}$ is nuclear. Thus T can be written in the form

$$T(x) = \sum_{k \geq 1} \lambda_k(x) e_k$$

with $a = \sum_{k\geq 1} \|\lambda_k\|\|e_k\| < \infty$. Fix an index j. For each $x \in T^{-1}(W_j)$, set $2r_{j,x} = p_U(Tx, \partial W_j) > 0$. Consider the Taylor expansion of g_j at $T(x)$:

$$g_j(Tx+z) = \sum_{n\geq 0} P_n g_j(Tx)(z)$$

for $\|z\| < 2r_{j,x}$, $z \in P_U$, where

$$P_n g_j(Tx)(z) = \frac{1}{2\pi i}\int_{|\lambda|=r_{j,x}} g_j(Tx+\lambda z)/\lambda^{n+1}\, d\lambda$$

for $\|z\| \leq 1$, $z \in P_U$. We have

$$\|P_n g_j(Tx)\| \leq M_j/(r_{j,x})^n \quad \text{with } M_j = \sup\{\|g_j(z)\| \colon z \in W_j\}.$$

Therefore

$$\begin{aligned}
&\sum_{n\geq 0}\sum_{k_1,\dots,k_n\geq 1} (\delta_{j,x})^n \|\lambda_{k_1}\|\|e_{k_1}\|\cdots\|\lambda_{k_n}\|\|e_{k_n}\| \\
&\qquad\qquad \times \|P_n g_j(Tx)(e_{k_1}/\|e_{k_1}\|, \dots, e_{k_n}/\|e_{k_n}\|)\| \\
&\quad \leq M_j \sum_{n\geq 0}(1/n!)(\delta_{j,x} n/r_{j,x})^n \left(\sum_{k\geq 1}\|\lambda_k\|\|e_k\|\right)^n \\
&\quad = M_j \sum_{n\geq 0}(1/n!)(\delta_{j,x} na/r_{j,x})^n < \infty \quad \text{with } \delta_{j,x} = r_{j,x}/2ae.
\end{aligned}$$

It follows that $g_jT|x+\delta_{j,x}B_V$, where B_V is the unit ball in P_V, can be written in the form

$$g_jT(z) = \sum_{\alpha\in\mathscr{B}} \lambda_\alpha(z-x)a_\alpha^{j,x}$$

for $z \in P_V$, $\|z-x\| < \delta_{j,x}$, where

$\mathscr{B} = \{\alpha \in (\mathbb{Z}^+)^{\mathbf{N}} \colon \alpha_j \neq 0 \text{ for only finitely many } j \in \mathbb{N}\}$,
$\lambda_\alpha = \lambda_{\alpha_1}\cdots\lambda_{\alpha_n}$, $a_\alpha^{j,x} = P_n g_j(Tx)(e_{\alpha_1}, \dots, e_{\alpha_n})$,
$\alpha = (\alpha_1, \dots, \alpha_n, 0, \dots,)$,

and

$$\sum_{n\geq 0}(\delta_{j,x})^n\|\lambda_\alpha\|\|a_\alpha^{j,x}\| < \infty.$$

Thus there exist $x_{j,k} \in T^{-1}(W_j)$, $\delta_{j,k} > 0$, $j, k = 1, 2, \dots,$ such that $\{x_{j,k} + \frac{1}{2}\delta_{j,k}B_V\}$ is an open cover of $T^{-1}(W_j)$ and

$$\sum_{n\geq 0}(\delta_{j,k})^n\|\lambda_\alpha\|\|a_\alpha^{j,k}\| < \infty, \qquad a_\alpha^{j,k} = a_\alpha^{j,x_{j,k}}$$

with $(g_jT)(x) = \sum_{\alpha\in\mathscr{B}} \lambda_\alpha(x-x_{j,k})a_\alpha^{j,k}$ for $x \in x_{j,k} + \delta_{j,k}B_V$.

For each $n \geq 1$, set $A_n = \{\alpha \in \mathscr{B} \colon \max \alpha_j \leq n\}$ and

$$S_n^{j,k}(x) = \sum_{\alpha\in A_n} \lambda_\alpha(x-x_{j,k})a_\alpha^{j,k}$$

for $x \in x_{j,k} + \delta_{j,k}B_V$.

Let $\varepsilon > 0$ be given. Take $m \in \mathbb{N}$ such that

$$\sum_{i>m} \|\lambda_i\| \|e_i\| < \varepsilon .$$

We have

$$\left\| \sum_{\alpha \in \mathscr{B}} \lambda_\alpha (x - x_{j,k}) a^{j,k} - S_m^{j,k}(x) \right\|$$
$$\leq \sum_{\max \alpha_j > m, n \geq 0} (\delta_{j,k})^n \|\lambda_{\alpha_1}\| \cdots \|\lambda_{\alpha_n}\| \|a_\alpha^{j,k}\|$$
$$\leq (M_j \varepsilon / a) \sum_{n \geq 0} (\delta_{j,k} n / r_{j,x_{j,k}})^n (1/n!) \left(\sum_{i \geq 1} \|\lambda_i\| \|e_i\| \right)^n$$
$$= (M_j \varepsilon / a) \sum_{n \geq 0} 1/n! (n/2a)^n .$$

Thus $S_n^{j,k} \to g_j T$ uniformly on $x_{j,k} + \delta_{j,k} B_V$ for $j, k \geq 1$ as $n \to \infty$. Applying the method of Bishop [2] to the sequence $\{S_n^{j,k}\}$, $j, k \geq 1$, we can find a sequence of disjoint 1-dimensional projections P_q^n in $F(B)$ such that

$$S_n^{j,k} = \sum_{q \geq 1} P_q^n S_n^{j,k} = \sum_{q \geq 1} h_q^{n,j,k} v_q^n$$

and

$$\|v_q^n\| = 1 , \quad \|P_q^n\| \leq 2^{\log_2^2 4q} , \quad P_q^n(v_q^n) = v_q^n$$
$$\sup_{n \geq 1} \sum_{q \geq 1} \|h_q^{n,j,k}\|_{x_{j,k} + \tilde{\delta}_{j,k} B_V} < \infty \quad \text{for } j, k \geq 1 ,$$

where $\tilde{\delta}_{j,k} = \frac{1}{2} \delta_{j,k}$ and $\|h_q^{n,j,k}\|_{x_{j,k} + \tilde{\delta}_{j,k} B_V}$ denotes the sup-norm of $h_q^{n,j,k}$ on $x_{j,k} + \tilde{\delta}_{j,k} B_V$.

Since $S(A) = B$, the map S induces a continuous linear map $\widetilde{S}$ from $E(A)$ onto $F(B)$. Thus the open mapping theorem gives a constant $C > 0$ such that for each (n, q) there exists $u_q^n \in E(A)$ for which $\widetilde{S}(u_q^n) = v_q^n$ with $\|u_q^n\| \leq C \|v_q^n\|$ for $n, q \geq 1$.

Set

$$\widetilde{S}_n^{j,k}(x) = \sum_{q \geq 1} h_q^{n,j,k} u_q^n$$

for $x \in x_{j,k} + \tilde{\delta}_{j,k} B_V$. Then

$$\sup_{n \geq 1} \{ \|\widetilde{S}_n^{j,k}(x)\| : x \in x_{j,k} + \tilde{\delta}_{j,k} B_V \}$$
$$\leq C \sup_{n \geq 1} \sum_{q \geq 1} \|h_q^{n,j,k}\|_{x_{j,k} + \tilde{\delta}_{j,k} B_V} < \infty .$$

Thus the sequence $\{\widetilde{S}_n^{j,k}\}_{n \geq 1}$ is bounded in $\mathscr{O}(x_{j,k} + \tilde{\delta}_{j,k} B_V , E(A))$, the space of holomorphic functions on $x_{j,k} + \tilde{\delta}_{j,k} B_V$ with values in $E(A)$ equipped with the compact-open topology. From the compactness of the canonical map $E(A) \to E$, we can assume that $\{\widetilde{S}_n^{j,k}\}_{n\geq 1}$ converges to $\widetilde{S}^{j,k}$ in $\mathscr{O}(x_{j,k} + \tilde{\delta}_{j,k} B_V, E)$ as $n \to \infty$ for all $j, k \geq 1$. Moreover we assume also that the sequences $\{v_q^n\}$, $\{P_q^n\}$, and $\{h_q^{n,j,k}\}$ converge to v_q, P_q and $h_q^{j,k}$ in F, $\operatorname{Hom}(F(B), F)$, the space of continuous linear maps from $F(B)$ into F, and $\mathscr{O}(x_{j,k} + \tilde{\delta}_{j,k} B_V)$, respectively, as $n \to \infty$ for all $j, k, q \geq 1$. On the other hand, from the relations

$$P_s^n v_q^n = 0 \quad \text{if } s \neq q \qquad \text{and} \qquad P_s^n v_s^n = v_s^n$$

we have

$$\begin{aligned} \lim_{n\to\infty} P_s^n \sigma_j g_i T &= \lim_{n\to\infty} P_s^n \sigma_j s_n^{j,k} \\ &= \lim_{n\to\infty} \sum_{q\geq 1} P_s^n \sigma_j h_q^{n,i,k} v_q^n = \lim_{n\to\infty} \sigma_j h_s^{n,i,k} v_s^n = \sigma_j h_s^{i,k} v_s . \end{aligned}$$

Similarly

$$\lim_{n\to\infty} P_s^n \sigma_i g_j T = \lim_{n\to\infty} P_s^n \sigma_i S_n^{j,l} = \sigma_i h_s^{j,l} v_s .$$

Hence

$$\sigma_j h_s^{i,k} = \sigma_i h_s^{j,l} \quad \text{for all } i, j, k, l \geq 1 .$$

This yields

$$\sigma_i \widetilde{S}^{j,k} = \sigma_j \widetilde{S}^{i,l}$$

on $(x_{j,k} + \tilde{\delta}_{j,k} B_V) \cap (x_{i,l} + \tilde{\delta}_{i,l} B_V)$ for all $i, j, k, l \geq 1$.

Thus the system $\{\widetilde{S}^{j,k}/\sigma_j\}$ defines an E-valued meromorphic function g on D such that $Sg = f$. The theorem is proved.

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