COHOMOLOGY OF NILPOTENT SUBALGEBRAS OF AFFINE LIE ALGEBRAS

A. FIALOWSKI

Abstract. We compute the cohomology of the maximal nilpotent Lie algebra of an affine Lie algebra \( \hat{g} \) with coefficients in modules of functions on the circle with values in a representation space of \( g \). These modules are not highest weight modules and are somewhat similar to the adjoint representation.

Introduction

Let \( g \) be a finite-dimensional semisimple Lie algebra, \( b \) a Borel subalgebra of \( g \), and \( n_+ \subset b \) the maximal nilpotent ideal of \( b \). The Bott-Kostant Theorem for Lie algebra cohomology is the following.

Theorem [K]. Let \( V \) be an irreducible representation of \( g \) with dominant highest weight and \( n \) a maximal nilpotent subalgebra of \( g \). Then \( \dim H^i(n; V) \) is equal to the number of elements of length \( i \) in the Weyl group of \( g \).

Consider the affine infinite-dimensional graded Lie algebra \( \hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \) corresponding to \( g \), with \( \hat{g}_i = g \otimes t^i \). There are at least two analogues of the above Theorem for affine algebras. The most direct analogue is the following: if \( V \) is an irreducible representation of the current algebra \( \hat{g} \) with dominant highest weight and \( \hat{n}_+ \) is a maximal nilpotent subalgebra of \( \hat{g} \), that is,

\[
\hat{n}_+ = (n_+ \otimes 1) \oplus (g \otimes t) \oplus (g \otimes t^2) \oplus \cdots,
\]

then \( \dim H^i(\hat{n}_+; V) \) is equal to the number of elements of length \( i \) in the Weyl group. This Theorem was proved by Garland in 1975 [G] and Garland and Lepowsky in 1976 (see [GL]). The proof is similar to that of the finite-dimensional case.

In this paper we present the proof of a different analogue of the Bott-Kostant Theorem obtained jointly with Feigin and announced in [FF]. Namely, we compute the cohomology of \( \hat{n}_+ \) with coefficients in modules of functions on the
circle $S^1$ with values in a representation space of $g$. These modules are not highest weight modules and are somewhat similar to the adjoint representation.

**RESULTS**

Let $V$ be a representation of $g$, $A$ a $C$-algebra, and $\varphi: C[t, t^{-1}] \to A$ a homomorphism. Let us define a representation of $\hat{g}$ on $V \otimes A$ by

$$(x \otimes f)(v \otimes a) = x(v) \otimes \varphi(f)a,$$

where $x \in \hat{g}$, $v \in V$, $f \in C[t, t^{-1}]$, and $a \in A$.

Consider two special cases for $A$ and $\varphi$:

(a) $A = C[t, t^{-1}]$ and $\varphi = \text{id}$. In this case denote the module $V \otimes A$ by $\hat{V}$. It consists of rational functions $C \to V$ that are regular outside the origin.

(b) $A = C$ and $\varphi(f) = f(1)$. In this case denote the module $V \otimes A$ by $V_1$.

Note that the map assigning to a function $C \to V$ its value at 1 defines a homomorphism $\hat{V} \to V_1$. The space $\hat{V}$ is endowed with an obvious module structure over the algebra $C[t, t^{-1}]$, and it is easy to see that multiplication by an element of $C[t, t^{-1}]$ is a $g$-endomorphism of the $\hat{g}$-module $\hat{V}$. Finally, note that $\hat{V}$ is a graded $\hat{g}$-module, that is, $\hat{V} = \bigoplus_{i \in \mathbb{Z}} \hat{V}_i$, with $\hat{V}_i = V \otimes t^i$.

First we will compute the cohomology of $\hat{n}_+$ with coefficients in $\hat{V}$. The Lie algebra $\hat{n}_+$ is a graded subalgebra of $\hat{g}$, and $\hat{V}$ is a graded $\hat{g}$-module and $\hat{n}_+$-module. Denote by $C^*(\hat{n}_+; \hat{V})$ the cochain complex of $\hat{n}_+$ with coefficients in the $\hat{n}_+$-module $\hat{V}$. The complex $C^*(\hat{n}_+; \hat{V})$ and the cohomology $H^*(\hat{n}_+; \hat{V})$ are graded by weights. To state this, we introduce the notation

$$C^*_m(\hat{n}_+; \hat{V}) = \bigoplus_{r \in \mathbb{Z}} \operatorname{Hom}((\wedge^q \hat{n}_+)_r, \hat{V}_{r+m}),$$

where

$$(\wedge^q \hat{n}_+)_r = \wedge^q \hat{n}_+ \cap \left( \bigotimes^q \hat{n}_+ \right)_r$$

and

$$\left( \bigotimes^q \hat{n}_+ \right)_r = \bigoplus_{r_1 + \cdots + r_q = r} ((\hat{n}_+)_{r_1} \otimes \cdots \otimes (\hat{n}_+)_{r_q}).$$

In this notation the grading is

$$C^*(\hat{n}_+; \hat{V}) = \bigoplus_{m \in \mathbb{Z}} C^*_m(\hat{n}_+; \hat{V})$$

and, similarly,

$$H^*(\hat{n}_+; \hat{V}) = \bigoplus_{m \in \mathbb{Z}} H^*_m(\hat{n}_+; \hat{V}).$$

**Lemma 1.** $H^*_m(\hat{n}_+; \hat{V}) \cong H^*(\hat{n}_+; V_1)$ for all $m \in \mathbb{Z}$.

**Proof.** The composition of mappings

$$C^*_m(\hat{n}_+; \hat{V}) \xrightarrow{i} C^*(\hat{n}_+; \hat{V}) \xrightarrow{s} C^*(\hat{n}_+; V_1),$$
where \( i \) is the embedding and \( s \) is induced by the homomorphism \( \hat{V} \to V_1 \), is obviously a complex isomorphism.

Moreover, the isomorphisms
\[
\hat{V}_i = V \otimes t^i \to V \otimes t^{i+1} = \hat{V}_{i+1}
\]
define a \( \hat{g} \)-isomorphism \( t : \hat{V} \to \hat{V} \) of degree 1, which generates an action of \( \mathbb{C}[t, t^{-1}] \) in \( \hat{V} \) and in \( H^*(\hat{n}_+; \hat{V}) \). Evidently, \( t \) maps \( H_{(m)}^*(\hat{n}_+; \hat{V}) \) isomorphically onto \( H_{(m+1)}^*(\hat{n}_+; \hat{V}) \). Hence we have

**Lemma 1'.** \( H^*(\hat{n}_+; \hat{V}) \cong \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} H^*(\hat{n}_+; V_1) \).

Now let us compute the cohomology \( H^*(\hat{n}_+; V_1) \). Introduce the subalgebra \( g[t] = g \otimes \mathbb{C}[t] \) of \( \hat{g} \). In the following we will identify \( g \) with \( g \otimes 1 \subset g[t] \). The Lie algebra \( \hat{n}_+ \) is embedded into \( g[t] \), and \( V_1 \) is naturally endowed with a \( g[t] \)-module structure.

**Theorem 1.** We have the following isomorphism of cohomology spaces:
\[
H^i(\hat{n}_+; V_1) \xrightarrow{\cong} \bigoplus_{p+q=i} H^p(\hat{n}_+; \mathbb{C}) \otimes H^q(g[t], g; V_1).
\]

**Proof.** We begin the proof, which will take most of this paper, by introducing two subalgebras of \( \hat{g} \). The first is
\[
\hat{g} = (t - 1)g \oplus (t - 1)^2g \oplus \cdots
\]
consisting of loops \( \varphi(t) \) which vanish at 1, and the second is
\[
\hat{n} = \hat{n}_+ \cap \hat{g}.
\]
Note that \( g[t] = \hat{n}_+ + \hat{g} \). Let \( G \) be the compact connected, simply connected Lie group, corresponding to the compact real form of \( g \). Next we need the following theorem.

**Theorem 2.** We have
\[
H^*(\hat{n}) \cong H^*(\hat{n}_+) \otimes H^*(\hat{g}) \otimes H^*(\Omega G),
\]
where \( \Omega G \) is the loop space of \( G \).

**Proof.** Since we have the embeddings

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\begin{tikzcd}
\hat{n}_+ \arrow[r, hook] \arrow[d, hook] & \\
\hat{n} \arrow[d, hook] \arrow[r, hook] & g[t] \\
\hat{g} \arrow[u, hook] \arrow[r, hook] & 
\end{tikzcd}
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with $\mathfrak{g} \mapsto \mathfrak{g}[t] \mapsto \mathfrak{g}$, we also have the diagram:

$$
\begin{array}{ccc}
C^*(\mathfrak{n}) & \rightarrow & C^*(\mathfrak{g}[t]) \\
\downarrow & & \downarrow \\
C^*(\mathfrak{g}) & \rightarrow & C^*(\mathfrak{n}_+) \\
\end{array}
$$

Consequently,

$$
C^*(\mathfrak{n}) = C^*(\mathfrak{n}_+) \otimes_{C^*(\mathfrak{g}[t])} C^*(\mathfrak{g}),
$$

where the tensor product is taken in the category of differential algebras. In such a situation there exists an Eilenberg-Moore spectral sequence, connecting these four differential algebras. Its second term is

$$
E_2 = \text{Tor}_{H^*(\mathfrak{g}[t])}(H^*(\mathfrak{n}_+), H^*(\mathfrak{g})),
$$

and its limit term is $H^*(\mathfrak{n})$. We know that $H^*(\mathfrak{g}[t]) \cong H^*(\mathfrak{g})$ (see, for example, [F]). On the other hand, since $H^*(\mathfrak{g})$ acts trivially on $H^*(\mathfrak{g})$ and also on $H^*(\mathfrak{n}_+)$, we conclude that the composition $H^*(\mathfrak{g}) \rightarrow H^*(\mathfrak{n}_+) \rightarrow H^*(\mathfrak{n})$ is trivial. So we have

$$
E_2 = \text{Tor}_{H^*(\mathfrak{g})}(H^*(\mathfrak{n}_+), H^*(\mathfrak{g})) \cong H^*(\mathfrak{n}_+) \otimes H^*(\mathfrak{g}) \otimes \text{Tor}_{H^*(\mathfrak{g})}(C, C).
$$

On the other hand,

$$
\text{Tor}_{H^*(\mathfrak{g})}(C, C) \cong H^*(\Omega G).
$$

Indeed, the cohomology algebra of $\mathfrak{g}$ with trivial coefficients coincides with the cohomology algebra of $G$, and by the Hopf Theorem it is commutative and free (see [S]). Using the computation of $\text{Tor}_A(C, C)$ for the free commutative algebra $A$ ([M, Proposition 7.3] and see also [A]) and the connection between the cohomology of $G$ and $\Omega G$, we get the isomorphism $\text{Tor}_A(C, C) \cong H^*(\Omega G)$:

$$
H^*(\mathfrak{g}) = \bigwedge^*(e_{a_1}, \ldots, e_{a_k}),
$$

where $e_{a_i} \in H^{a_i}$. So with the mapping $G \rightarrow \Omega G$ we have

$$
\text{Tor}_{\bigwedge^*}(e_{a_1}, \ldots, e_{a_k})(C, C) = S^*(c_{a_1-1}, \ldots, c_{a_k-1}),
$$

where $\deg c_{a_i-1} = a_i - 1$. The generators of the cohomology are the homotopy groups. To complete the proof of Theorem 2 we will need the next proposition.

**Proposition 1.** The spectral sequence degenerates, namely, its second term $E_2$ coincides with the limit term $E_\infty$.

**Proof.** We shall indicate explicit cocycles of $C^*(\mathfrak{n})$ which represent the generators of $E_2$. For this we apply the continuous cohomology theory. Let $n(0, 1)$ be the Lie algebra of infinitely differentiable functions $f: [0, 1] \rightarrow \mathfrak{g}$ such that $f(0) \in n$ and $f(1) = 0$. Denote by $C^*_c(0, 1)$ the complex of cochains of $n(0, 1)$, continuous in the $C^\infty$-topology. Let $\alpha$ be a generator of $H^*(\mathfrak{g})$ and...
a cochain representing \( \alpha \). For \( p \in [0, 1] \) denote by \( \varphi_p \) the homomorphism \( n \to g \), "the value at \( p \)":

\[
\varphi_p((t-1)g_1, (t-1)^2g_2, \ldots) = \sum (p-1)^m g_m.
\]

Let \( \alpha_p = \varphi_p \alpha, \alpha_p \in C_*^*(0, 1) \). Choose \( \bar{\alpha} \) in such a way that \( \alpha_0 = \alpha_1 = 0 \). Let \( p \neq 0, 1 \); then we can define the cochain \( \frac{\partial \alpha}{\partial x}(p) \), where \( x \) is the coordinate on \([0, 1]\). It is shown in [F] that \( \frac{\partial \alpha}{\partial x}(p) \) is a coboundary: \( \frac{\partial \alpha}{\partial x}(p) = \delta \omega(p) \), where \( \delta \) is the differential in \( C_*^*(0, 1) \). Indeed, let \( K_p (p \neq 0) \) be the cochain complex of \( n \) with support at \( p \). It is proved in the same paper that the cohomology of \( K_p \) is isomorphic to \( H^*(g) \). The complex \( K_p \) is a \( W_1 \)-module, where \( W_1 \) is the Lie algebra of formal vector fields at the point \( p \). But \( H^*(g) \) is finite dimensional and \( W_1 \) has no nontrivial finite-dimensional representations. We conclude that if \( \nu \in K_p \) and \( \delta \nu = 0 \) then \( \frac{\partial \nu}{\partial x} \) is the differential of some other cocycle \( \bar{\nu} \in K_p \).

This means that

\[
\alpha_p - \alpha_q = \delta \int_q^p \omega(x) \, dx.
\]

In particular, \( \delta \int_0^1 \omega(x) \, dx = 0 \). Suppose that \( \alpha' = \int_0^1 \omega(x) \, dx \). The cochain \( \alpha' \) represents a nontrivial cohomology class of \( n \).

The Lie algebras \( n_+ \) and \( g = (t-1)g \oplus (t-1)^2g \oplus \cdots \) are graded. Similarly, the cochain complexes are also graded. Note that the cochain complex \( K_0 \) of \( n(0, 1) \) with support in \( 0 \) is isomorphic to \( \bigoplus_i C_i^*(n_+) \) and the cochain complex \( K_1 \) with support in \( 1 \) is isomorphic to \( \bigoplus_i C_i^*(g) \). It follows from this that the cohomologies of \( K_0 \) and \( K_1 \) are isomorphic to \( H^*(n_+) \) and \( H^*(g) \), respectively.

Recall that \( H^*(g) \) is isomorphic to the free graded commutative algebra with generators \( \xi_1, \xi_2, \ldots \), with \( \deg \xi_k = 2k + 1 \). Using the above construction, let us assign to each \( \xi_i \) a representative cocycle \( \xi_i' \).

Lemma 2. The space \( H^*(n) \) is generated by the cohomology classes of cochains of the form \( u \wedge v \wedge P(\xi_1', \xi_2', \ldots) \), where \( u \in K_0 \) and \( v \in K_1 \) are cocycles corresponding to the elements of \( H^*(n_+) \) and \( H^*(g) \), respectively, and \( P \) is an arbitrary polynomial with generators \( \xi_1', \xi_2', \ldots \).

The proof of Lemma 2 follows from the construction above for continuous cohomology (a similar argument in a more difficult situation was used in [FR]). In particular, we have an explicit construction of cochains, representing the generators of \( E_2 \) in the proof of Theorem 2, surviving until \( E_{\infty} \). Thus, Theorem 2 is proved.

We now return to complete the proof of Theorem 1. We want to prove the isomorphism

\[
H^*(g[t], g; V_1) \otimes H^*(n_+; C) \cong H^*(n_+; V_1).
\]

Consider the Serre-Hochschild spectral sequence associated with the algebra \( n_+ \), its ideal \( n \), and the module \( V_1 \). The Lie algebra \( n \) acts on \( V_1 \) trivially. The second term of this spectral sequence is

\[
E_2^{ij} = H^i(\bar{n}_+; n); H^j(\bar{n}; V_1)) = H^i(\bar{n}_+; n); H^j(\bar{n}; C) \otimes V_1).
\]

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But by Theorem 2, this is the same as

\[ H^i \left( \frac{\mathfrak{n}_+}{\mathfrak{n}}, \bigoplus_{p+q+r=j} H^p(\mathfrak{n}_+) \otimes H^q(\mathfrak{g}) \otimes H^r(\Omega G) \otimes V_1 \right) \].

Since \( \mathfrak{n}_+ / \mathfrak{n} \cong \mathfrak{g} \), we get that \( E_2 \) is then isomorphic to

\[ H^*(\mathfrak{g}, \mathbb{C}) \otimes [H^*(\mathfrak{n}_+) \otimes H^*(\mathfrak{g}) \otimes H^*(\Omega G) \otimes V_1]^{\mathfrak{g}} \],

where \([ \quad ]^{\mathfrak{g}} \) denotes the invariant space.

Let us note the following facts:

(a) \( \mathfrak{g} \) acts on \( H^*(\mathfrak{n}_+) \) trivially (this action is extended by the projection \( \mathfrak{n}_+ \to \mathfrak{g} \) to the canonical action of \( \mathfrak{n}_+ \), and a Lie algebra acts trivially on the cohomology of itself).

(b) \( \mathfrak{g} \) acts trivially on \( H^*(\Omega G) = \text{Tor}_{H^*(\mathfrak{g})}(\mathbb{C}, \mathbb{C}) \).

These imply that

\[ E_2^{ij} = H^j(\mathfrak{g}) \otimes \left( \bigoplus_{p+q+r=j} (H^p(\mathfrak{n}_+) \otimes H^q(\Omega G) \otimes [H^r(\mathfrak{g}) \otimes V_1])^{\mathfrak{g}} \right) \]

and

\[ [H^*(\mathfrak{g}) \otimes V_1]^{\mathfrak{g}} = H^*(\mathfrak{g}[t], \mathfrak{g} ; V_1). \]

The differentials act in the following way.

(i) Differentials on \( H^*(\mathfrak{n}_+) \otimes H^*(\mathfrak{g}[t], \mathfrak{g} ; V_1) \) are zero. We have the map

\[ H^*(\mathfrak{n}_+) \otimes H^*(\mathfrak{g}[t], \mathfrak{g} ; V_1) \to H^*(\mathfrak{n}_+ ; V_1). \]

Thus, elements of the left side survive in \( E_\infty \).

(ii) Differentials on

\[ H^*(\mathfrak{g}) \otimes H^*(\Omega G) = \bigwedge^* \{e_{a_i}\} \otimes S^*\{c_{a_i-1}\} \]

map the generators of the algebra \( H^*(\Omega G) \) into the generators of \( H^*(\mathfrak{g}) \) (the differential maps \( c_{a_i-1} \mapsto e_{a_i} \)).

Consider the Serre path fibration \( EG \to G \). Since the paths are contractible, \( H^0(EG) = \mathbb{C} \) and \( H^i(EG) = 0 \) for \( i > 0 \). In addition, we have \( H^*(\mathfrak{g}) = H^*(G) \), so

\[ H^*(G) \otimes H^*(\Omega G) \] converges to \( H^*(EG) \cong \mathbb{C} \).

Then it follows that the spectral sequence converges to \( H^*(\mathfrak{n}_+) \otimes H^*(\mathfrak{g}[t], \mathfrak{g} ; V_1) \). Thus our spectral sequence is the product of \( H^*(\mathfrak{n}_+) \otimes H^*(\mathfrak{g}[t], \mathfrak{g} ; V_1) \) with the spectral sequence of the Serre path fibration \( EG \to G \). This completes the proof of Theorem 1.

**Theorem 3.** \( H^i(\hat{\mathfrak{n}}_+ ; \hat{\mathfrak{g}}) \cong \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} H^{i-1}(\mathfrak{n}_+) \) for any nonnegative integer \( i \).

**Proof.** Let \( V = \mathfrak{g} \) as in Lemma 1'. Then Lemma 1' implies that \( H^i(\hat{\mathfrak{n}}_+ ; \hat{\mathfrak{g}}) \) is a free \( \mathbb{C}[t, t^{-1}] \)-module of rank equal to \( \dim H^i(\mathfrak{n}_+ ; V_1) \).

The cohomology of \( \mathfrak{n}_+ \) with trivial coefficients is known (see [GL]). Using this result, it is not difficult to find the cohomology \( H^*(t\mathfrak{g}[t]) \) of the algebra \( t\mathfrak{g}[t] = (\mathfrak{g} \otimes t) \oplus (\mathfrak{g} \otimes t^2) \oplus \cdots \) (see also [G] and [GL]; the proof in [GL] is basically the same as the proof of the cohomology result of \( \mathfrak{n}_+ \) with trivial coefficients). We need only the following fact. The space \( H^*(t\mathfrak{g}[t]) \) is a \( \mathfrak{g} \)-module
and $\text{Hom}_g(g; H^i(tg[t])) = 0$ if $i \neq 1$ and is equal to $\mathbb{C}$ if $i = 1$ (see [L]). Since $H^i(g[t], g; V) \cong \text{Hom}_g(V; H^i(tg[t]))$, this gives us that $H^i(g[t], g; g) = 0$ for $i \neq 1$ and is equal to $\mathbb{C}$ for $i = 1$. After this, it is enough to apply Theorem 1 to find that $H^i(\hat{n}_+; V_1) = H^{i-1}(\hat{n}_+)$. 

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**REFERENCES**


**E-mail address**: fialowsk@math.ucdavis.edu