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A LERAY-SCHAUDER TYPE THEOREM FOR APPROXIMABLE MAPS

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ABSTRACT. We present an elementary proof of a Leray-Schauder type theorem for approximable set-valued maps. Our theorem generalizes many results for convex as well as nonconvex maps. Our argument is not based on a homotopy invariance property but, quite surprisingly, on a matching theorem of Ky Fan on closed covers of convex sets.

All topological vector spaces in this paper are assumed to be real Hausdorff spaces. Given a set X, $\mathscr{P}(X)$ denotes the family of all nonempty subsets of X. In what follows, X and Y are two subsets of two topological vector spaces E and F respectively. The boundary, the interior, and the convex hull of a subset X of E are denoted by ∂X, $\operatorname{int} X$, and $\operatorname{co} X$ respectively. For brevity, locally convex topological vector spaces are called locally convex spaces.

Definitions. Let $\Phi\colon X \to \mathscr{P}(Y)$ be a map.

(1) Φ is said to be *upper semicontinuous* (u.s.c.) on X if the set $\{x \in X | \Phi(x) \subset V\}$ is open in X whenever V is an open subset of Y.

(2) Φ is said to be compact if $\Phi(X)$ is relatively compact in Y.

(3) Given two open neighborhoods U and V of the origins in E and F respectively, a (U, V)*-approximative continuous selection* of Φ is a continuous function $s\colon X \to Y$ satisfying

$$s(x) \in (\Phi[(x + U) \cap X] + V) \cap Y \quad \text{for every } x \in X.$$

(4) Φ is said to be *approximable* if its restriction $\Phi|K$ to any compact subset K of X admits a (U, V)-approximative continuous selection for every open neighborhoods U and V of the origins in E and F respectively.

Note that Φ is approximable if and only if for any compact subset K of X every open neighborhood of the graph of $\Phi|K$ contains the graph of a continuous function.

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Examples. Let $\Phi\colon X \to \mathscr{P}(Y)$ be a u.s.c. map.

(1) (*Convex case*) If X is a topological space, Y is a convex subset in a locally convex space F; and if the values of Φ are convex, then Φ is approximable [C, Theorem I].

(2) (*Nonconvex cases*) Assume that X is contained in a topological vector space E and that Y is contained in a locally convex space F. Then Φ admits a (U, V)-approximative continuous selection for any open neighborhoods of the origins U and V in E and F respectively, provided one of the following conditions is satisfied:

(2.1) X is a compact ANR, Y is an ANR, and the values of Φ are compact and contractible [BD, Corollary 4.3] ([MC, Lemma] for the finite-dimensional case).

(2.2) X is a separable metric space, $Y = L^p$, and the values of Φ are decomposable in the sense of [HU] [BC, Theorem 2].

(2.3) X is a compact ANR, and the values of Φ are compact and ∞-proximally connected in Y [BD, Corollary 4.5] ([GGK, Proposition] for the metrizable case).

The concept of ∞-proximally connectedness was introduced in [D]. Note that if Y is an ANR, then each contractible subspace of Y is ∞-proximally connected. R_δ-sets in ANR's are also ∞-proximally connected (see [GGK]).

It was recently shown that if X is a convex subset of a locally convex space and (2.3) is satisfied, then Φ is approximable [BOT].

For more properties and examples of approximable maps we refer to [B1, BC, BD, BOT, GGK, GL].

The key result of our paper is the following.

Proposition. *Let X be a closed subset of a topological vector space E such that $0 \in \operatorname{int} X$, and let $\Phi\colon X \to \mathscr{P}(E)$ be a u.s.c. approximable map. If Φ is compact then, given any symmetric convex open neighborhood V of 0 in E such that $2V$ is contained in $\operatorname{int} X$, one of the following properties is satisfied*:

- (α) *There exists $x_V \in X$ with $x_V \in \Phi(x_V) + V$ (V-approximative fixed point).*
- (β) *There exists $(\lambda_V, x_V) \in (0, 1) \times \partial X$ such that $\hat{x}_V \in \lambda_V \Phi(\hat{x}_V) + V$ for some $\hat{x}_V \in (x_V + V) \cap X$ (V-approximative invariant direction).*

Some sufficient conditions for the existence of V-approximative fixed points were presented in [I]. The proof of our Proposition is based on a finite-dimensional approximation property for compact approximable maps and on a matching theorem of Fan on closed covers of convex sets.

Lemma 1 [B1, Proposition 2.11]. *Let X be a subset of a topological vector space E, Y a subset of a topological vector space F, and $\Phi\colon X \to \mathscr{P}(Y)$ a u.s.c. approximable compact map. Then for any open convex neighborhood V of the origin in F there exists a u.s.c. map $\Phi_V\colon X \to \mathscr{P}(F)$ satisfying the following two properties*:

- (a) *Φ_V is an approximable map from X into the convex hull of some finite subset N of Y.*
- (b) *For every $x \in X$, $\Phi_V(x) \subset \Phi(x) + V$.*

Lemma 2 [F, Theorem 2]. *Let X be a convex subset in a topological vector space E, and let $\{F_i|i \in I\}$ be a finite family of relatively closed subsets of X such that $\bigcup\{F_i|i \in I\} = X$. Then for any collection $\{y_i|i \in I\}$ of points of X indexed by the same set I there exists a nonempty subset J of I such that the convex hull of $\{y_i|i \in J\}$ contains a point of the intersection $\bigcap\{F_i|i \in J\}$.*

Proof of the Proposition. Let $\Phi_V: X \to \mathscr{P}(E)$ be a finite-dimensional approximation of Φ provided by Lemma 1; that is, Φ_V verifies

$\Phi_V(X) \subset C = \operatorname{co} N$, where N is a finite subset of E, and
$\Phi_V(x) \subset \Phi(x) + \frac{1}{4}V$ for all $x \in X$.

We may assume with no loss of generality that $0 \in C$ (otherwise, we replace C by $\operatorname{co}\{C \cup \{0\}\}$). Let $s: X \cap C \to C$ be a $(\frac{1}{4}V, \frac{1}{4}V)$-approximative continuous selection of the map

$$\Phi_V|X \cap C: X \cap C \to \mathscr{P}(C);$$

that is, s satisfies

for every $x \in X \cap C$ there exists $\hat{x} \in (x + \frac{1}{4}V) \cap X$ such that
$s(x) \in \Phi_V(\hat{x}) + \frac{1}{4}V$.

For every $x \in X \cap C$ let $y = s(x)$. Thus we obtain an open cover $\mathscr{V} = \{s^{-1}(y + \frac{1}{4}V)|y \in s(X \cap C)\}$ of $X \cap C$. Let $\{O_i|i \in \widehat{I}\}$ be a finite open subcover of $\mathscr{V}$, and let $\{F_i|i \in \widehat{I}\}$ be a finite closed cover of $X \cap C$ with $F_i \subset O_i$ for each $i \in \widehat{I}$. Define a finite closed cover $\{F_i|i \in I = \widehat{I} \cup \{i_0\}\}$ of C by adding the closed set $F_{i_0} = \overline{E \backslash X} \cap C$.

For each $i \in I$ let us choose a point $y_i \in C$ as follows:

if $i \in \widehat{I}$, choose y_i so that $F_i \subset s^{-1}(y_i + \frac{1}{4}V)$, and if $i = i_0$,
choose y_i to be 0.

All hypotheses of Lemma 2 are satisfied. Thus there exists a subset J of I such that

$$\bigcap\{F_i|i \in J\} \cap \operatorname{co}\{y_i|i \in J\} \neq \varnothing.$$

Let x_V be in this intersection. Two possibilities can now occur.

Possibility 1. The index i_0 does not belong to J. In this case, $x_V \in F_i \subset s^{-1}(y_i + \frac{1}{4}V)$ for each $i \in J$ and, by symmetry of V, $y_i \in s(x_V) + \frac{1}{4}V$ for each $i \in J$. Since V is convex, $x_V \in s(x_V) + \frac{1}{4}V$. On the other hand, $s(x_V) + \frac{1}{4}V \subset \Phi_V(\hat{x}_V) + \frac{1}{2}V \subset \Phi(\hat{x}_V) + \frac{3}{4}V$ for some $\hat{x}_V \in (x_V + \frac{1}{4}V) \cap X$. Therefore, $\hat{x}_V \in \Phi(\hat{x}_V) + V$ and (i) is verified.

Possibility 2. The index i_0 belongs to J. In this case, note that J cannot reduce to the singleton $\{i_0\}$; otherwise $x_V = 0 \in (\overline{E \backslash X} \cap C) \cap \operatorname{int} X$, which is impossible. Therefore, x_V belongs to $F_i \subset s^{-1}(y_i + \frac{1}{4}V)$ for each $i \in J$, $i \neq i_0$, and $x_V \in \overline{E \backslash X} \cap C$; that is, $x_V \in \partial X$. The real number $\lambda_V = 1 - \lambda_{i_0} = \sum_{i \in J \backslash \{i_0\}} \lambda_i$ is strictly positive, and $x_V/\lambda_V = z = (\sum_{i \in J \backslash \{i_0\}} \lambda_i y_i)/\lambda_V$ is a convex combination. Since, for $i \neq i_0$, $x_V \in s^{-1}(y_i + \frac{1}{4}V)$, that is, $y_i \in s(x_V) + \frac{1}{4}V$; by convexity of V, $z \in s(x_V) + \frac{1}{4}V$, and finally $z = x_V/\lambda_V \in \Phi(\hat{x}_V) + \frac{3}{4}V$ for some $\hat{x}_V \in (x_V + \frac{1}{4}V) \cap X$. Thus we have proved the existence of $(\lambda_V, x_V) \in (0, 1) \times \partial X$ such that $\hat{x}_V \in \lambda_V \Phi(\hat{x}_V) + V$ for some $\hat{x}_V \in (x_V + V) \cap X$. □

Our main result is a Leray-Schauder type theorem which, in the convex case, has found many applications in the theory of differential equations and differential inclusions. Its proof is based on Proposition and on the following fixed-point property.

Lemma 3 [B1, Lemma 4.1]. *Let X be a regular topological space, A a subspace of X, and $\Phi: A \to \mathscr{P}(X)$ a u.s.c. map with closed values. Assume that there exists a cofinal subfamily of covers $\{\mathscr{U}\}$ in the family of all open covers of $\overline{\Phi(X)}$ in X such that, for each cover $\mathscr{U}$, Φ has a V-approximative fixed point for some $V \in \mathscr{U}$. Then Φ has a fixed point.*

Theorem. *Let X be a closed subset of a locally convex space E such that $0 \in \operatorname{int} X$ and $\Phi: X \to \mathscr{P}(E)$ a compact u.s.c. approximable map with closed values. If Φ is fixed-point free, then it satisfies the following Leray-Schauder condition:*

$$\textit{there exists } (\lambda, x) \in (0, 1) \times \partial X \textit{ such that } x \in \lambda\Phi(x).$$

Proof. If Φ is fixed-point free, then by Lemma 3 there exists an open neighborhood of the origin U in E such that $2U$ is contained in $\operatorname{int} X$ and Φ has no U-approximative fixed point. Hence by our Proposition Φ has a V-invariant direction for all open neighborhoods V of the origin in E contained in U; that is, there exists $(\lambda_V, x_V) \in (0, 1) \times \partial X$ such that $\hat{x}_V \in \lambda_V\Phi(\hat{x}_V) + V$ for some $\hat{x}_V \in (x_V + V) \cap X$. Therefore, the map $\Psi: X \to \mathscr{P}(E)$ defined by $\Psi(x) = \bigcup\{\lambda\Phi(x); 0 \leq \lambda \leq 1\}$ for all $x \in X$ being a compact u.s.c. map with closed values has a V-approximative fixed point $\hat{x}_V$ for each open neighborhood of the origin V in E contained in U. Thus by Lemma 3 the map Ψ has a fixed point x; that is $x \in \lambda\Phi(x)$ for some $\lambda \in [0, 1]$. It is easily seen that λ must belong to $(0, 1)$ and x to ∂X. □

Corollary. *Let X be a closed subset of a locally convex space E such that $0 \in \operatorname{int} X$ and $\Phi: X \to \mathscr{P}(E)$ a compact u.s.c. map and closed values. If Φ is fixed-point free, then it satisfies the following Leray-Schauder condition:*

$$\textit{there exists } (\lambda, x) \in (0, 1) \times \partial X \textit{ such that } x \in \lambda\Phi(x),$$

provided that one of the following is satisfied:

(i) *The values of Φ are convex* [M, *Theorem* 16.1].
(ii) *The values of Φ are contractible.*
(iii) *The values of Φ are decomposable.*
(iv) *The values of Φ are ∞-proximally connected* [GGK, *partie* 1, *Corollaire* 1].

Remarks. (1) Our Theorem generalizes [B2, Theorem 5] where a similar argument was used in the proof under assumption (i).

(2) We mention that finite compositions of approximable maps are also approximable (see [B1, GL]). Hence Corollary is valid for compact u.s.c. maps with closed values which are finite compositions of maps satisfying one of the conditions (i)–(iv).

(3) Corollary is in fact valid for acyclic maps [G, Theorem 6]. The proof there is based on a homotopy invariant, namely, topological transversality. Let us point out that the proofs of [M, Theorem 16.1] and [GGK, Corollaire] are based on the homotopy invariance of the index and the topological degree.

(4) In the case where X is convex and condition (i) is satisfied, an elementary proof of the result above is presented in [R]. In fact, it is proven there that this result holds true even when, instead of being compact, the map Φ is condensing and the space E is quasi-complete. The convexity assumption on X is removed for the single-valued case in [R, Part II].

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