A CONNECTION BETWEEN WEAK REGULARITY AND THE SIMPLICITY OF PRIME FACTOR RINGS

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Dedicated to the memory of Professor Pere Mennal

Abstract. In this paper, we show that a reduced ring $R$ is weakly regular (i.e., $I^2 = I$ for each one-sided ideal $I$ of $R$) if and only if every prime ideal is maximal. This result extends several well-known results. Moreover, we provide examples which indicate that further generalization of this result is limited.

Throughout this paper $R$ denotes an associative ring with identity. All prime ideals are assumed to be proper. The prime radical of $R$ and the set of nilpotent elements of $R$ are denoted by $\mathcal{P}(R)$ and $N(R)$, respectively. The connection between various generalizations of von Neumann regularity and the condition that every prime ideal is maximal will be investigated. This connection has been investigated by many authors [2, 3, 5, 7, 12, 14]. The earliest result of this type seems to be by Cohen [3, Theorem 1]. Storrer [12] was able to provide the following result: If $R$ is a commutative ring then the following are equivalent: (1) $R$ is $\pi$-regular; (2) $R/\mathcal{P}(R)$ is regular; and (3) all prime ideals of $R$ are maximal ideals. Fisher and Snider extended this result to P.I. rings [5, Theorem 2.3]. On the other hand, Chandran generalized Storrer’s result to duo rings [2, Theorem 3]. Next Hirano generalized Chandran’s result to right duo rings [7, Corollary 1]. More recently the result was generalized to bounded weakly right duo rings by Yao [14, Theorem 3].

As a corollary of our main result, we show that if $R/\mathcal{P}(R)$ is reduced (i.e., $N(R) = \mathcal{P}(R)$) then the following are equivalent: (1) $R/\mathcal{P}(R)$ is weakly regular; (2) $R/\mathcal{P}(R)$ is right weakly $\pi$-regular; and (3) every prime ideal of $R$ is maximal. This result generalizes Hirano’s result for right duo rings. A further consequence of our main result is that if $R$ is reduced then $R$ is weakly regular if and only if every prime factor ring of $R$ is a simple domain. This result can be compared to the well-known fact that when $R$ is reduced, then $R$ is von Neumann regular if and only if every prime factor ring of $R$ is a division ring. We conclude our paper with some examples which illustrate and delimit our results.

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Definition 1. (1) A ring $R$ is right (left) weakly regular if $I^2 = I$ for each right (left) ideal $I$ of $R$, equivalently $x \in xRxR$ ($x \in RxRx$) for every $x \in R$. $R$ is weakly regular if it is both left and right weakly regular [11]. Note right (left) weakly regular rings are also called right (left) fully idempotent rings.

(2) A ring $R$ is called $\pi$-regular if for every $x \in R$ there exists a natural number $n = n(x)$, depending on $x$, such that $x^n \in x^nRx^n$.

(3) A ring $R$ is right (left) weakly $\pi$-regular if for every $x \in R$ there exists a natural number $n = n(x)$, depending on $x$, such that $x^n \in x^nRx^n$ ($x^n \in Rx^nRx^n$). $R$ is weakly $\pi$-regular if it is both right and left weakly $\pi$-regular [6]. Every $\pi$-regular ring, biregular ring (including simple rings), and right duo ring satisfying d.c.c. on principal ideals is right weakly $\pi$-regular.

Definition 2. A ring $R$ is 2-primal if and only if $P(R) = N(R)$ [1].

Definition 3. An ideal $I$ of $R$ is completely prime if $xy \in I$ implies either $x \in I$ or $y \in I$ where $x, y \in R$. Also $I$ is called completely semiprime if $x^2 \in I$ implies $x \in I$.

Lemma 4. A ring $R$ is 2-primal if and only if every minimal prime ideal is completely prime.

Proof. See [13, Proposition 1.11].

Lemma 5. If $R$ is a 2-primal ring and $R/P(R)$ is right weakly $\pi$-regular, then every prime ideal of $R$ is maximal.

Proof. Let $P$ be a prime ideal of $R$. Then there exists a minimal prime ideal $X \subseteq P$ which is completely prime by Lemma 4. Let $\bar{R} = R/X$. Then $\bar{R}$ is a right weakly $\pi$-regular domain. Let $a$ be a nonzero element in $\bar{R}$. There exists a positive integer $k$ such that $a^k(y - 1) = 0$, where $y \in aRa^k\bar{R}$. Hence $R/X$ is a simple ring. Thus $X$ is a maximal ideal and so is $P$.

A ring $R$ is called strongly $\pi$-regular if for every $x$ in $R$ there exists a natural number $n = n(x)$, depending on $x$, such that $x^nR = x^{n+1}R$. By Dischinger [4], this condition is left-right symmetric.

From Lemma 5 and [5], we obtain the following result which appeared in [7].

Corollary 6. Let $R$ be a 2-primal ring. Then the following conditions are equivalent:

1. $R$ is strongly $\pi$-regular.
2. $R$ is $\pi$-regular.
3. Every prime factor ring of $R$ is a division ring.

Proof. Obviously (1) implies (2). Also by Lemma 5, (2) implies (3). Now by [5], since every prime factor ring is strongly $\pi$-regular by condition (3), $R$ is strongly $\pi$-regular. Thus (3) implies (1).

Lemma 7. If $S$ is a completely semiprime ideal of $R$ and $x_1x_2 \cdots x_n \in S$, then $x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)} \in S$, where $\sigma$ is any permutation of $\{1, \ldots, n\}$.

Proof. See [9] and [10].

Recall that a ring is reduced if there is no nonzero nilpotent element.
Theorem 8. Let \( R \) be a reduced ring. Then the following conditions are equivalent:

1. \( R \) is weakly regular.
2. \( R \) is right weakly \( \pi \)-regular.
3. Every prime ideal of \( R \) is maximal.

Proof. Clearly (1) implies (2). Lemma 5 shows that (2) implies (3). So we will assume that every prime ideal of \( R \) is maximal and show that \( R \) is weakly regular. Observe that in a reduced ring every minimal prime ideal is completely prime by Lemma 4 or [8]. Now since every prime ideal is maximal, then every prime ideal is completely prime. Let \( a \) be any nonzero element in \( R \). If \( RaR = R \), then \( a = a1 \in aRaR \). Thus we may assume \( RaR \neq R \). Then \( RaR \) is contained in a maximal ideal which is also a prime ideal. Let \( T \) be the union of all prime ideals which contain \( a \). Let \( S = R \setminus T \). Since every prime ideal is completely prime, \( S \) is a multiplicatively closed set. Let \( F \) be the multiplicatively closed system generated by the set \( \{a\} \cup S \).

Now we assert that \( 0 \in F \). Suppose this were not true, then partial order the collection of ideals disjoint with \( F \) by set inclusion. By Zorn's lemma, we get an ideal \( M \) which is maximal disjoint with \( F \). Then \( M \) is a prime ideal and so a maximal ideal by hypothesis. Since \( a \notin M \), there exist \( p \in M \) and \( c \in RaR \) such that \( p + c = 1 \). It follows that \( p \notin T \). Thus \( p \in S \subseteq F \), which implies \( p \in F \cap M = \emptyset \), a contradiction. Thus \( 0 \in F \), so

\[0 = a^{n_1}s_1a^{n_2}s_2 \cdots a^{n_t}s_t,\]

where \( s_i \in S \), and we may assume the integers \( n_1, n_2, \ldots, n_t \) are positive. Then, using Lemma 7 and the fact that \( R \) is reduced, there exists \( s \in S \) such that \( as = 0 \). Observe a proper ideal cannot contain both \( a \) and \( s \) (otherwise a prime ideal would contain both of them which would contradict the definition of \( S \) and \( T \)). Hence \( RaR + RsR = R \). Let \( a_0 \in RaR \) and \( s_0 \in RsR \) such that \( a_0 + s_0 = 1 \). Therefore \( aa_0 + as_0 = a \). Now using Lemma 7 and the fact that \( \{0\} \) is a completely semiprime ideal, \( aRsR = 0 \). Thus \( a = a_0 \in aRaR \). Similarly, \( a = a_0 a \in RaRa \). Consequently \( R \) is weakly regular.

Corollary 9. Let \( R \) be a 2-primal ring. The following conditions are equivalent:

1. \( R/P(R) \) is weakly regular.
2. \( R/P(R) \) is right weakly \( \pi \)-regular.
3. Every prime ideal of \( R \) is maximal.

Proof. Clearly (1) implies (2). Lemma 5 shows that (2) implies (3). So we will assume that every prime ideal of \( R \) is maximal. Then every prime ideal of \( R/P(R) \) is maximal. Since \( R \) is 2-primal, \( R/P(R) \) is reduced. Thus by Theorem 8, \( R/P(R) \) is weakly regular.

Corollary 10 [7, Corollary 1]. Let \( R \) be a right (or left) duo ring. Then \( R \) is \( \pi \)-regular if and only if every prime ideal is maximal.

Proof. Since any prime ideal is completely prime in a right (or left) duo ring, \( R \) is 2-primal. Assume \( R \) is \( \pi \)-regular. Then \( R \) is weakly \( \pi \)-regular, and so \( R/P(R) \) is weakly \( \pi \)-regular. Thus by Corollary 9, every prime ideal is maximal.

Conversely, by Corollary 9, \( R/P(R) \) is weakly regular. Since \( R \) is right duo, \( R/P(R) \) is strongly regular. Then each prime factor ring of \( R \) is also strongly regular.
regular. Hence by Theorem 2.1 of [5], \( R \) is both left and right \( \pi \)-regular. Consequently \( R \) is a \( \pi \)-regular ring.

Hirano [7] has shown that for a P.I. ring the concepts of right weak \( \pi \)-regularity, \( \pi \)-regularity, and strong \( \pi \)-regularity are equivalent. With this in mind, one can see that Corollary 9 is analogous to Theorem 2.3 of [5]. However these results are distinct in that Corollary 9 can be applied to a simple domain which is not a division ring and hence is not a P.I. ring. When \( R \) is reduced, it is well known that \( R \) is von Neumann regular if and only if every prime factor ring of \( R \) is a division ring. It is interesting to compare this fact with the following corollary.

**Corollary 11.** Let \( R \) be a reduced ring. Then \( R \) is right (and so left) weakly regular if and only if every prime factor ring of \( R \) is a simple domain.

Finally we provide two examples.

**Example 12.** By Fisher and Snider [5], it was shown that a ring \( R \) is strongly \( \pi \)-regular if and only if \( R/P(R) \) is strongly \( \pi \)-regular. But this fact does not hold for the case when \( R \) is right weakly \( \pi \)-regular. Indeed there is a 2-primal ring \( R \) such that \( R/P(R) \) is weakly regular but \( R \) is neither left nor right weakly \( \pi \)-regular. Assume that \( W_1[F] \) is the first Weyl algebra over a field \( F \) of characteristic zero. Recall \( W_1[F] = F[\mu, \lambda] \), the polynomial ring with indeterminates \( \mu \) and \( \lambda \) with \( \lambda \mu = \mu \lambda + 1 \). Then \( W_1[F] \) is a simple Noetherian domain. Now let \( R \) be the ring

\[
\begin{pmatrix}
W_1[F] & W_1[F] \\
0 & W_1[F]
\end{pmatrix}.
\]

Consider the following element in \( R \):

\[
a = \begin{pmatrix} \mu & \lambda \\ 0 & 0 \end{pmatrix}.
\]

Then it can be easily checked that \( a^k \) is not in \( a^k R a^k R \) for any positive integer \( k \). So \( R \) is not right weakly \( \pi \)-regular. Also it can be checked that \( R \) is not left weakly \( \pi \)-regular.

Now the prime radical \( P(R) \) of \( R \) is

\[
\begin{pmatrix}
0 & W_1[F] \\
0 & 0
\end{pmatrix}.
\]

So \( R/P(R) \cong W_1[F] \oplus W_1[F] \), which is weakly regular, and hence it is weakly \( \pi \)-regular. Furthermore we may check that \( P(R) = N(R) \) and hence \( R \) is a 2-primal ring.

**Example 13.** In Corollary 9, the condition "\( R \) is 2-primal" is not superfluous. As in Example 12, let \( W = W_1[F] \) be the first Weyl algebra over a field \( F \) of characteristic zero. Now let

\[
R = \{(s_k)_{k=1}^{\infty} | s_k \in \text{Mat}_2(W) \text{ is eventually a constant upper triangular matrix}\},
\]

which is a subring of \( \prod \text{Mat}_2(W) \), where \( \text{Mat}_2(W) \) denotes the full ring of
2-by-2 matrices over $W$. Then it can be easily checked that the ring $R$ is a semiprime ring. Now our claim is that every prime ideal is maximal. Let $P$ be a prime ideal of $R$.

Case 1. Assume that the $n$th component of all elements of $P$ is zero for some $n$. Let $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$, where 1 is in the $n$th component. Let $x \in R$ such that $x$ has zero in its $n$th component. Then $e_nRx = 0$ and so $x$ is in $P$. Therefore,

$$P = \{ (s_k)_{k=1}^\infty \in R | s_n = 0 \}$$

and thus $P$ is maximal.

Case 2. Assume that for any $n$, there exists an element of $P$ with a nonzero entry in its $n$th component. In this case, $\bigoplus_{k=1}^\infty \text{Mat}_2(W) \subseteq P$. Now let

$$I_0 = \left\{ (s_k)_{k=1}^\infty \in R | s_k = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \text{ eventually for some } b \in W \right\}.$$

Then $I_0^2 = \bigoplus_{k=1}^\infty \text{Mat}_2(W) \subseteq P$ and so $I_0 \subseteq P$. Let

$$e = \left\{ (s_k)_{k=1}^\infty | s_k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } k \right\}$$

and

$$f = \left\{ (s_k)_{k=1}^\infty | s_k = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ for all } k \right\}.$$

Then $eRf \subseteq I_0 \subseteq P$ and so either $e$ is in $P$ or $f$ is in $P$. Now without loss of generality, assume that $e$ is in $P$. Then

$$I_1 = \left\{ (s_k)_{k=1}^\infty | s_k = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \text{ eventually for some } a \text{ and } b \text{ in } W \right\}$$

is contained in $P$. So if $y \in R \setminus P$, then we may assume that $y = (y_k)_{k=1}^\infty$ with

$$y_k = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

eventually for some $a, b, c$ in $W$. Then in this situation, $P + RyR = R$ and so $P$ is maximal. Furthermore, consider the element $s = (s_k)_{k=1}^\infty$ in $R$ such that

$$s_k = \begin{pmatrix} \mu & \lambda \\ 0 & 0 \end{pmatrix}$$

for all $k$. Then for any $m$, $s^m \notin s^mRs^mR$ and hence $R$ is not right weakly $\pi$-regular. Therefore, in our Corollary 9, the condition "$R$ is 2-primal" is not superfluous. Also note that there is a prime von Neumann regular ring which is not simple. So, in our Corollary 9, without the hypothesis "$R$ is 2-primal", there is no relation between $R/P(R)$ is right weakly $\pi$-regular and the fact that every prime ideal is maximal.

Recall that a ring is of bounded index $k$ of nilpotency if $a^k = 0$ for every nilpotent element $a$. Therefore, a reduced ring is a ring of bounded index 1 of nilpotency. So one might suspect the possibility of a generalization of Theorem 8 to the case of semiprime rings of bounded index of nilpotency. But Example 13 nullifies this possibility because $R$ is a semiprime ring of bounded index 2 of nilpotency.
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ADDED IN PROOF

Recently, K. Beidar and R. Wisbauer notified the authors that they have announced a result generalizing Theorem 8 in Strongly semiprime modules and rings, Research Math. Survey, Moscow Math. Soc. 48 (1993), 161–162. Also E. P. Armendariz has generalized Theorem 8 in work he announced in On rings with all prime ideals maximal, Abstracts Amer. Math. Soc. 14 (1993), 732.

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