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ON LIFTING THE HYPERELLIPTIC INVOLUTION

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ABSTRACT. Let W_p stand for a compact Riemann surface of genus p.

(1) Let W_q be hyperelliptic, and let n be a positive integer. Then there exists an unramified covering of n sheets, $W_p \to W_q$, where W_p is hyperelliptic.

(2) Let $W_{2n+1} \to W_2$ be an unramified Galois covering with a dihedral group as Galois group, and let n be odd. Then W_{2n+1} is elliptic hyperelliptic (bi-elliptic).

(3) Let $W_4 \to W_2$ be an unramified non-Galois covering of three sheets. Then W_4 is hyperelliptic.

1. INTRODUCTION

Let W_p stand for a compact Riemann surface of genus p. All Riemann surfaces in this paper will be compact.

Concerning automorphisms of Riemann surfaces there are two well-known results.

(1) If $W_p \to W_q$ is a smooth (unramified) abelian covering of a hyperelliptic W_q, then the hyperelliptic involution lifts to W_p [5].

(2) If $W_3 \to W_2$ is a (necessarily smooth 2-sheeted) covering, then W_3 is hyperelliptic. (This latter result seems to have first appeared in a paper by Enriques [2]. For an interesting proof see Farkas [3].)

In this paper we will discuss two questions.

(1) W_q is hyperelliptic, to which (not necessarily Galois) smooth coverings of W_q does the hyperelliptic involution lift?
(2) Which of these coverings is itself hyperelliptic?

The kinds of answers given here appear in the abstract.

If W_q is hyperelliptic ($q \geq 2$), then there is a 2-sheeted covering $W_q \to \mathbf{P}^1$ branched over $2q+2$ points in $\mathbf{P}^1$, $a_1, a_2, \ldots, a_{2q+2}$. Let $X = \mathbf{P}^1 - \{a_1, a_2, \ldots, a_{2q+2}\}$. Let γ_i be the homotopy class in $\pi_1(X, \cdot)$ of a curve which "circles" a_i. Denote $\pi(X, \cdot)$ by $\mathscr{F}$, and let F_q be the subgroup of index 2 in $\mathscr{F}$ of words in the γ_i's where the sum of the exponents is even. In the correspondence between subgroups of $\mathscr{F}$ and coverings of X, F_q corresponds to W_q punctured at the points above the a_i's. A smooth covering

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$W_p \to W_q$ corresponds to a subgroup, F_p, of F_q which contains all the γ_i^2's and their conjugates in $\mathscr{F}$. We will call such a subgroup *smooth*. A smooth Galois covering $W_p \to W_q$ to which the hyperelliptic involution lifts corresponds to the kernel of a homomorphism $\mu: \mathscr{F} \to G'$ where G' is a finite group containing a subgroup G of index 2 where

(i) the kernel of the composition $\mathscr{F} \xrightarrow{\mu} G' \to G'/G$ is F_q; that is, $\mu(\gamma_i) \in G' - G$ for all i; and

(ii) $\mu(\gamma_i)^2 = e$ for all i.

If F_p is the kernel of μ, then the Galois group of the covering $W_p \to W_q$, denoted $\mathscr{G}(W_p, W_q)$, is isomorphic to G. Any involution T in $\mathscr{G}(W_p, \mathbf{P}^1)$ with a fixed point will be a lift of the hyperelliptic involution, and $\mathscr{G}(W_p, \mathbf{P}^1)$ will be the semidirect product of $\langle T \rangle$ and $\mathscr{G}(W_p, W_q)$.

If $W_p \to W_q$ is a smooth non-Galois covering, the hyperelliptic involution lifts to W_p if there is $\eta \in \mathscr{F} - F_q$ so that $F_p^\eta = F_p$ and $\eta^2 \in F_p$. We have the following array of groups:

$$\begin{array}{ccc} F_p & \longrightarrow & F_q \\ \searrow & & \searrow \\ \langle F_p, \eta \rangle & \longrightarrow & \langle F_q, \eta \rangle = \mathscr{F} \end{array}$$

The corresponding array of coverings is (W_r corresponds to $\langle F_p, \eta \rangle$):

$$\begin{array}{ccc} W_p & \longrightarrow & W_q \\ \searrow & & \searrow \\ W_r & \longrightarrow & \mathbf{P}^1 \end{array}$$

Conjugating F_q (resp. F_p) by η corresponds to the action of the (resp. lift of the) hyperelliptic involution on W_q (resp. W_p).

Now we shall discuss more systematically the two results mentioned in the second paragraph of this paper.

Lemma 1. *If $W_p \to W_q$ is a smooth abelian covering and W_q is hyperelliptic, then the hyperelliptic involution lifts to W_p to be an involution which need not be unique or hyperelliptic. If $g \in \mathscr{G}(W_p, W_q)$ and h is a lift of the hyperelliptic involution, then $h^{-1}gh = g^{-1}$. Thus if $\mathscr{G}(W_p, W_q) = \langle g \rangle$, then $\langle g, h \rangle$ is a dihedral group, D_n, of order $2n$ where n is the order of g.*

Since we will use Lemma 1 later we now discuss dihedral groups of automorphisms. If $D_n = \langle V, R \rangle$, $V^2 = R^n = VRVR = e$, then VR^x is conjugate to VR^y if and only if $x - y$ is even. If n is even, there are two conjugacy classes of reflections, and if n is odd, there is only one.

Suppose W_p admits a group of automorphisms isomorphic to D_n. Let p_1 be the genus of $W_p/\langle V \rangle$; let p_2 be the genus of $W_p/\langle VR \rangle$; let p_R be the genus of $W_p/\langle R \rangle$; and let p_0 be the genus of W_p/D_n. Then [1]

$$p + 2p_0 = p_1 + p_2 + p_R. \tag{1}$$

If n is odd, then $p_1 = p_2$ since conjugate automorphism groups yield conformally equivalent quotients. (*Notation.* Let Z_n stand for the cyclic group of order n.)

Lemma 2. *Let* $W_p \to W_q$ *be a smooth cyclic covering of* n *sheets where* n *is odd and* W_q *is hyperelliptic. Then the covering* $W_p \to W_q \to \mathbf{P}^1$ *is Galois with* $\mathcal{G}(W_p, \mathbf{P}^1) \cong D_n$. *If* V *is a reflection in* D_n, *then the genus of* $W_p/\langle V\rangle$ *is* $(n-1)(q-1)/2$.

Proof. Apply formula (1) since we know $p = n(q-1)+1$, $p_R = q$, $p_0 = 0$, and $p_1 = p_2$. Q.E.D.

Lemma 3. *Let* W_3 *be a (necessarily smooth) 2-sheeted covering of* W_2. *Then* W_3 *is hyperelliptic.*

Proof. Apply Lemma 1 and formula (1) with $n = 2$. Q.E.D.

2. Smooth hyperelliptic coverings

Given that W_q is hyperelliptic we consider the problem of a smooth covering W_p of W_q again being hyperelliptic. If the covering is Galois, the answer was given by Maclachlan [7] and Horiuchi [6].

Lemma 4. *If* $W_p \to W_q$ *is a smooth Galois covering where* W_p *is hyperelliptic, then the order of* $\mathcal{G}(W_p, W_q)$ *divides* 4.

If, however, we do not require the covering to be Galois, there is no restriction on the number of sheets in the covering.

Theorem 1. *Let* W_q *be a hyperelliptic Riemann surface, and let* n *be a positive integer. Then there exists a smooth* n*-sheeted covering* $W_p \to W_q$ *and* W_p *is hyperelliptic.*

Proof. If $n = 2$, then the result is well known [4, 7], and if n is a power of 2, the result follows by induction. It remains to prove the theorem for n odd.

We now define a smooth Galois covering $W_{2p-1} \to W_q$ with Galois group D_n, n odd, and W_q is hyperelliptic in the context of the fourth paragraph of this paper. To do this we define a homomorphism

$$\mu: \mathcal{F} \to Z_2 \times D_n \qquad (\cong D_{2n} \text{ since } n \text{ is odd})$$

as follows. Let $Z_2 = \langle C\rangle$, $D_n = \langle V, R\rangle$ as above. Then $\langle C\rangle$ is the center of D_{2n}, and $\langle CV, R\rangle$ is also isomorphic to D_n.

Let

$$\begin{aligned} \mu(\gamma_i) &= C \quad \text{for } i = 1, 2, \ldots, 2q-2, \\ \mu(\gamma_{2q-1}) &= \mu(\gamma_{2q}) = CV, \\ \mu(\gamma_{2q+1}) &= \mu(\gamma_{2q+2}) = CVR. \end{aligned}$$

μ then extends to a homomorphism onto D_{2n}, and $\mu(\gamma_i)^2 = e$ for all i. The kernel of the composition

$$\mathcal{F} \to Z_2 \times D_n \to \langle C\rangle$$

corresponds to W_q (branch points have been filled in), and the kernel of μ, F_{2p-1}, corresponds to a smooth covering $W_{2p-1} \to W_q$ with Galois group isomorphic to D_n ($\cong F_q/F_{2p-1}$). The genus of W_{2p-1} is $2n(q-1)+1$. $\mathcal{G}(W_{2p-1}, \mathbf{P}^1)$ is isomorphic to $Z_2 \times D_n$. In this group of automorphisms let $Z_2' = \langle C'\rangle$ and let $D_n' = \langle V', R'\rangle$. (We use primes to distinguish the automorphisms on W_{2p-1} from the elements of the abstract group D_{2n}.)

The central involution has branch points above a_i, $i = 1, 2, \dots, 2q-2$, so the ramification for the covering $W_{2p-1} \to W_{2p-1}/\langle C' \rangle$ is $2n(2q-2)$. By the Riemann-Hurwitz formula the genus of $W_{2p-1}/\langle C' \rangle$ is one.

Assume $\mathcal{G}(W_{2p-1}, W_q)$ is $\langle V', R' \rangle$. The other D_n is $\langle C'V', R' \rangle$ which contains n reflections, all of whose fixed points lie over the a_i for $i = 2q-1, \dots, 2q+2$.

Thus each such reflection has $4(2n)/n$ $(= 8)$ fixed points. By the Riemann-Hurwitz formula the genus of $W_{2p-1}/\langle C'V' \rangle$ is $p-2$.

Now consider the four group $\{V', C', C'V', e\}$ $(= H)$. The genera of the quotients with respect to these three involutions are respectively p, 1, and $p-2$. By formula (1) the genus of W_{2p-1}/H is zero. Thus $W_{2p-1}/\langle V' \rangle$ is hyperelliptic and is a smooth n-sheeted covering of W_q. Q.E.D.

3. Galois coverings

Lemma 5. *Let $W_p \to W_q$ be a smooth non-Galois covering where W_q is hyperelliptic. Suppose the hyperelliptic involution, T, lifts to W_p. Then the hyperelliptic involution lifts to the Galois closure, $\widetilde{W}$, for the covering $W_p \to W_q$.*

Proof. Let T' be the lift of T to W_p. Then we have the following array of Riemann surfaces:

$$\begin{tikzcd} \widetilde{W} \arrow[r] & W_p \arrow[r] \arrow[d] & W_q \arrow[d] \\ & W_p/\langle T' \rangle \arrow[r] & W_q/\langle T \rangle \ (= \mathbf{P}^1) \end{tikzcd}$$

Referring to subgroups of $\mathcal{F}$ (as in the fourth paragraph of this paper), we have the following array of subgroups of $\mathcal{F}$ $(\widetilde{F} = \bigcap_{\sigma \in F_q} F_p^\sigma)$:

$$\begin{tikzcd} \widetilde{F} \arrow[r] & F_p \arrow[r] \arrow[d] & F_q \arrow[d] \\ & F' \arrow[r] & \mathcal{F} \end{tikzcd}$$

where F_q, F_p, F', and $\widetilde{F}$ correspond to W_q, W_p, $W_p/\langle T' \rangle$, and $\widetilde{W}$.

There is an element $\eta \in \mathcal{F} - F_q$ so that $F' = \langle F_p, \eta \rangle$, $F_p^\eta = F_p$, and $\eta^2 \in F_p$. Also $\langle F_q, \eta \rangle = \mathcal{F}$, and since F_q is normal in $\mathcal{F}$, $F_q^\eta = F_q$. We now show that $\widetilde{F}^\eta = \widetilde{F}$. Since $\widetilde{F} = \bigcap_{\sigma \in F_q} F_p^\sigma$,

$$\widetilde{F}^\eta = \bigcap_{\sigma \in F_q} F_p^{\sigma\eta} = \bigcap_{\sigma} F_p^{\eta(\eta^{-1}\sigma\eta)} = \bigcap_{\sigma} F_p^{\eta^{-1}\sigma\eta} = \bigcap_{\sigma} F_p^\sigma = \widetilde{F}$$

since $\sigma \to \eta^{-1}\sigma\eta$ is a permutation of F_q.

Thus $\widetilde{F}$ is normal in $\mathcal{F}$. The Galois covering $\widetilde{W} \to W_p \to W_q \to \mathbf{P}^1$ has branch points of multiplicity 2 corresponding to involutions which are lifts of the hyperelliptic involution of W_q. Q.E.D.

As an aside from the main arguments of this paper, we consider the situation $W_p \to W_q \xrightarrow{2} \mathbf{P}^1$, a Galois covering where $W_p \to W_q$ is smooth, so the hyperelliptic involution lifts. If H is a subgroup of $\mathscr{G}(W_p, W_q)$, we ask for conditions on H which assure that the hyperelliptic involution lifts to W_p/H.

Let $G = \mathscr{G}(W_p, \mathbf{P}^1)$, and let $K = \mathscr{G}(W_p, W_q)$. Thus $G/K \cong Z_2$. For $H \subset K$ let $N_K(H)$ be the normalizer of H in K and let $N_G(H)$ have similar meaning.

Definition. A subgroup H of K will be called *inner* if whenever we have an automorphism ϕ of K, there is $\sigma \in K$ so that $\phi(H) = H^\sigma$.

Examples are Sylow subgroups, Hall subgroups of solvable groups, and any subgroup of a group with a trivial outer automorphism group.

Theorem 2. *Let $W_p \to W_q \xrightarrow{2} \mathbf{P}^1$ be a Galois covering where $W_p \to W_q$ is smooth. Suppose H is an inner subgroup of K.*

Then the hyperelliptic involution lifts to $W_p/N_K(H)$.

Proof. Consider the following array of coverings:

$$\begin{tikzcd}[column sep=small]
 & & & W_p/K\ (= W_q) \arrow[dr] & \\
W_p \arrow[r] & W_p/H \arrow[r] & W_p/N_K(H) \arrow[ur] \arrow[dr] & & W_p/G\ (= \mathbf{P}^1) \\
 & & & W_p/N_G(H) \arrow[ur] &
\end{tikzcd}$$

and the corresponding array of groups:

$$\begin{tikzcd}[column sep=small]
 & & & K \arrow[dr] & \\
\langle e\rangle \arrow[r] & H \arrow[r] & N_K(H) \arrow[ur] \arrow[dr] & & G \\
 & & & N_G(H) \arrow[ur] &
\end{tikzcd}$$

Since K is normal in G, $N_K(H)$ is normal in $N_G(H)$. Also $N_G(H) \cap K = N_K(H)$. It now suffices to show that $G = KN_G(H)$. To do this let $\sigma \in G$. Since $\theta \to \theta^\sigma$ is an automorphism of K and H is inner, $H^\sigma = H^\eta$ for some $\eta \in K$. Thus $\sigma\eta^{-1} = n$ where $n \in N_G(H)$ and $\sigma = n\eta$. Q.E.D.

The reader may also wish to check that if H is the 2-Sylow subgroups of K, then the hyperelliptic involution lifts to W_p/H.

4. $q = 2$

We now assume $q = 2$ and use some ideas in the proof of Theorem 1.

Theorem 3. *Let $W_{2n+1} \to W_2$ be a smooth dihedral covering where n is odd. Then W_{2n+1} is elliptic hyperelliptic. If $G(W_{2n+1}, W_2) = \langle V', R'\rangle$ $(\cong D_n)$, then $W_{2n+1}/\langle V'\rangle$ is hyperelliptic.*

Proof. $W_{2n+1}/\langle R'\rangle$ $(= W_3)$ is a 2-sheeted covering of W_2 and is hyperelliptic. $W_{2n+1} \to W_3$ is a cyclic covering, so the hyperelliptic involution on W_3 lifts to

W_{2n+1}. We apply Lemma 1 to see that the $2n$-sheeted covering $W_{2n+1} \to W_3 \to \mathbf{P}^1$ is dihedral. Denote this Galois group $\langle U', R'\rangle$. We have the following array of coverings:

$$\begin{tikzcd}
 & & W_2 \arrow[dr] & \\
W_{2n+1} \arrow[r] & W_3 \arrow[r] \arrow[ur] \arrow[dr] & W_1 \arrow[r] & \mathbf{P}^1 \\
 & & \mathbf{P}^1 \arrow[ur] &
\end{tikzcd}$$

We see that the entire array is Galois with Galois group isomorphic to $Z_2 \times D_n$ ($\cong D_{2n}$ since n is odd). Then central element, C', of order 2 in D_{2n} is unique. We may assume $C' = U'V'$ by replacing V' by $V'R'^{\alpha}$ for suitable α.

We are now in the situation considered in the proof of Theorem 1 with $q = 2$, $X = \mathbf{P}^1 - \{a_1, \dots, a_6\}$, and $\mu\colon \pi_1(X, \cdot) \to D_{2n}$ where $D_{2n} = \langle C, V, R\rangle$ and $CV = U$. Since $\mu(\gamma_i)$ has order 2 for all i and all $\mu(\gamma_i)$'s lie outside one of the D_n's in D_{2n}, say $\langle V, R\rangle$, the possibilities for $\mu(a_i)$ are C and CVR^{α}. Since $\langle CV, R\rangle \cong D_n$ at least one of the $\mu(a_i)$'s must be C. It follows that two of the $\mu(a_i)$'s must be C. The argument in Theorem 1 now completes the proof. Q.E.D.

Corollary 1. *Let $W_4 \to W_2$ be an unramified* 3*-sheeted covering.*

(a) *If the covering is Galois, then W_4 is elliptic hyperelliptic.*
(b) *If the covering is not Galois, then W_4 is hyperelliptic.*

Proof. (a) Follows from Lemma 2 with $n = 3$, $q = 2$.

(b) The Galois closure of $W_4 \to W_2$ is $W_7 \to W_4 \to W_3$ where $\mathcal{G}(W_7, W_2) = D_3$. The result follows from Theorem 3. Q.E.D.

Corollary 2. *Let G be a group of order n with a nonnormal subgroup, H, of index* 3. *Suppose K is a subgroup of H, normal in H, so that H/K is abelian and $\bigcap_{g\in G} g^{-1}Kg = \langle e\rangle$. Suppose finally that $W_{n+1} \to W_2$ is an unramified Galois covering with Galois group isomorphic to G. Then the hyperelliptic involution lifts to W_{n+1}.*

Proof. Consider the array of coverings

$$W_{n+1} \to W_{n+1}/K \to W_{n+1}/H \to W_2\ (= W_{n+1}/G).$$

By Corollary 1 W_{n+1}/H is hyperelliptic. The covering $W_{n+1}/K \to W_{n+1}/H$ is abelian, so the hyperelliptic involution lifts to W_{n+1}/K. The covering $W_{n+1} \to W_{n+1}/H$ is the Galois closure in G of $W_{n+1}/K \to W_{n+1}/H$. By Lemma 5 the hyperelliptic involution lifts to W_{n+1}. Q.E.D.

Examples of such groups are S_4, the symmetric group of order 24, and the dicyclic group of order 12.

In the light of Lemma 3 and Corollary 1 one might consider 4-sheeted coverings $W_5 \to W_2$. The monodromy group, M, for this covering is a subgroup of S_4. If $M = Z_2 \times Z_2$, Z_4, D_4, or S_4, then previous results show that the hyperelliptic involution lifts to the Galois closure of $W_5 \to W_2$, and it follows by Theorem 2 that it lifts to W_5. (If $M = S_4$, apply Corollary 2.) If $M = A_4$, there is, unfornuately, a counterexample which shows that the hyperelliptic involution need not lift. In cases where it does lift W_5 is hyperelliptic or elliptic hyperelliptic.

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